THE

MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.
WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2.
AND BOMBAY

Vol. XII., No. 172. OCTOBER, 1924. 2s. 6d. Net.

CONTENTS.

PAGE

THE MANCHESTER UNIVERSITY SUMMER SCHOOL OF MATHEMATICS AT BANGOR. PROF. S. CHAPMAN, F.R.S., - - - - - - - - 185

MATHEMATICAL PHYSICS IN UNIVERSITY AND SCHOOL. PROF. H. T. H. PIAGGIO, D.Sc., - - - - - - - - - - - - - - 187

NOTE ON THE PARALLEL POSTULATE. A. A. KRISHNASWAMI AYYANGAR, M.A.; L.T., - - - - - - - - - - - - - - - 191

PARALLELISM AND SIMILARITY. D. K. PICKEN, M.A., - - - - - 195

MATHEMATICAL NOTES (727-742). J. W. BROOKS, M.A.; PROF. F. CAJORI, PH.D.; MAJOR C. H. CHEPMELL, R.E.; R. HAMILTON DICK, M.A.; W. J. DOBBS, M.A.; W. L. FERRAR, M.A.; S. L. GREEN, M.A.; P. L. HALL, M.A.; A. LODGE, M.A.; R. F. MUIRHEAD, M.A.; PROF. E. H. NEVILLE, M.A.; H. E. PIGGOTT, M.A.; PROF. H. J. PRIESTLEY, M.A.; A. ROBSON, M.A.; PROF. D. M. Y. SOMMERVILLE, D.Sc.; C. O. TUCKEY, M.A., - - - - - - - - - - - - - - 202

REVIEWS. W. HOPE-JONES, B.A.; A. LODGE, M.A.; E. A. MILNE, M.A.; R. M. MILNE, M.A.; PROF. E. H. NEVILLE, M.A.; PROF. H. T. H. PIAGGIO, D.Sc.; W. F. SHEPPARD, D.Sc.; PROF. G. N. WATSON, F.R.S., - - - - - - - - - - - - - - - 217

GLEANINGS FAR AND NEAR (258-269), - - - - - - - - - 190

THE LIBRARY, - - - - - - - - - - - - - - - 227

BOOKS RECEIVED, ETC., - - - - - - - - - - - - - i

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."

Change of Address should be notified to a Secretary. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

F.

T

Tr
fo
co
It
pu
tio
co
to
tiv
fu
ar
th
M
ar
E
ar
it
ar
st
m
so
al
fa
de

in
N
m
u
d
in
S
c
a

THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc., AND PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON:

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

AND BOMBAY.

VOL. XII. OCTOBER, 1924. No. 172.

THE MANCHESTER UNIVERSITY SUMMER SCHOOL OF MATHEMATICS AT BANGOR.

BY PROF. S. CHAPMAN, F.R.S.

THE desirability of affording facilities, to mathematical teachers and others, for post-graduate studies in mathematics during the summer vacation, was commented on by the writer in an article in the March issue of this *Gazette*. It was there proposed that a " Summer School " should be organized for this purpose by teachers themselves, acting through the Mathematical Association, and after the publication of the article the suggestion was more formally communicated to the Association through the Manchester branch. It proved to be unlikely, however, that the matter could be taken up by the representative body, whose members met rather infrequently, and who were already fully occupied in other business of the Association, in sufficient time to make arrangements for such a School during the present year. The proposal was therefore brought before the Extra-Mural Committee of the University of Manchester, who agreed to the experiment being made under their auspices, and voted a grant for organizing and advertising the project. The Board of Education were next approached, and they agreed to recognize the School and to make a limited grant to cover a possible deficit on the cost of working it. Provisional arrangements were proceeded with meanwhile, and in April and the early part of May circulars announcing the School and inviting students to register were distributed to the heads of the mathematical departments of nearly every secondary, grammar and high school, and many technical schools and colleges, throughout England and Wales. The School was also advertised in various educational journals. The response was satisfactory, thirty-two students being registered, whereas twenty was the number decided upon as the minimum required if the School was to be held.

The organization of the School was similar, on the whole, to that proposed in the *Gazette* article already referred to. The School was held at Bangor, North Wales, from Monday, August 25, to Saturday, September 6. Arrangements were made with the authorities of University College, Bangor, for the use of lecture rooms, and for hostel accommodation for such students as desired it. Twenty-four members of the School (students and staff) resided in one of the hostels, which formed the social meeting-place for the whole School. The success of the School owed much to the courtesy and friendly co-operation of the Bangor University Authorities (who placed the Library at the disposal of members), and particularly to Miss Davis, the Principal

Warden, and her staff. Bangor also proved excellent as a centre for walks and excursions by rail, sea, and motor, so that serious study could be combined with agreeable recreation.

Three courses of study were arranged, of which each student chose one only; the three courses were, indeed, held simultaneously. Each course, comprising twenty lectures of one hour each (given from 9.30 to 11.30, with a ten minutes' interval, on ten mornings), was planned as far as possible with reference to the acquirements of those who had registered for it. The three courses were on Higher Geometry (by Mr. H. W. Richmond, M.A., F.R.S., of King's College, Cambridge), Theory of Functions (by Prof. L. J. Mordell, F.R.S., of Manchester University) and the Elements of the Theory of Relativity (by the writer). The numbers of students attending these respective courses were thirteen, nine, and ten. The tuition fee was three guineas; one or two students obtained grants from their local education authorities towards their expenses in attending the School. The fee for board and lodging at the hostel was five guineas for the period of twelve days.

The course on Higher Geometry began with a sketch (based on a quotation from Henri Poincaré) of the characteristics of each of the three periods in the history of the subject—pre-Descartes, post-Descartes, and the last hundred years. The great advantage of combining pure and analytical methods, which in the earlier part of geometry can be developed side by side, was urged: the distinction between projective, metrical and affine geometry was explained and illustrated. Finally a special theorem (Pascal's) was taken, and an account was given of its history, its gradual growth into a figure of vast complexity, its later simplification by Cremona, and the discovery of its close relation to a variety of other much-discussed figures in space or in a plane.

The course on Theory of Functions, of a complex variable, commenced with the definition of real and complex numbers by means of sequences. The convergency of sequences was discussed, and the method of defining functions of real and complex variables, geometrical illustrations being given. The simpler functions naturally suggest power series, and the properties of the latter lead on to the question of the uniform convergence of series of functions. Complex integration and Cauchy's theorem were explained and applied to Taylor's and Laurent's expansions and to the evaluation of definite integrals.

The course on Relativity was confined to the special theory. It began with a historical survey of the theories and experiments which led up to Einstein's formulation of the special principle of relativity in 1905. The Lorentz-Einstein transformation was developed, and the relativist form of the principles of conservation of momentum and of mass or energy. Minkowski's treatment of the subject, leading to the conception of 4-vectors, was described, and illustrated by the momentum-energy 4-vector. The equations of motion were given in Planck's form, and were applied to the motion of an electron round an atomic nucleus. The bearing of this problem on Sommerfeld's theory of the fine structure of spectral lines was indicated.

The students numbered nineteen men and thirteen women, and were drawn from a widespread area, as, for example, Workington in the north, Anglesey in Wales, and, in the south, Folkestone, Portsmouth and Maidstone. Two were students still in residence at universities, two were on the staffs of university colleges, while the majority were engaged in secondary schools. Mr. W. C. Fletcher, H.M. Chief Inspector for Secondary Schools, was present both as a member of the School and in an official capacity.

It is intended to organize a similar school for next year. Plans and detailed particulars will be ready early in the New Year and correspondence or enquiries concerning it should be addressed to Miss D. Withington, The University, Manchester.

S. Chapman.

MATHEMATICAL PHYSICS IN UNIVERSITY AND SCHOOL.

By Prof. H. T. H. Piaggio, M.A., D.Sc.

A striking article under this title appeared in *Nature* of 10th May, 1924. The anonymous writer deplored the present position of mathematical physics in England, and suggested that the mathematical course at Cambridge was now much inferior to what it was in the days of Kelvin, Stokes and Maxwell, in that the physical aspect had been almost crowded out, leaving pure mathematics predominant. As a result, it was declared, there was only one man out of all those who had taken the Mathematical Tripos in the last forty years who had any claim to be recognised for his researches in mathematical physics apart from astronomy and relativity, in which it was admitted that excellent work had been done. The remainder of the article was a strong plea for the inclusion of physics in every mathematical course.

It is possible that the writer rather overstated his case. Anyone familiar with recent research will immediately think of at least five British mathematical physicists. A little reflection or a glance at *Science Abstracts* will recall the names of several more. The predominance of pure mathematics in the Tripos papers has been exaggerated. In 1923 nearly half the questions dealt with applied mathematics. Of course many of these consisted of the traditional type of statical or dynamical question of negligible physical interest. On the other hand, there were several questions on quantum theory and one on wireless telegraphy. Strangely enough, there was none on relativity. The papers were arranged in such a way that those who wished could choose quite two-thirds of their questions from the applied side. On the most advanced papers candidates were expected to confine themselves to a small number of questions all dealing with the subject in which they had specialised. In fact, the Cambridge Tripos seems to be more flexible and more in touch with modern ideas than any other mathematical examination in this country.

But although the case may have been overstated, it can hardly be denied that the position of mathematical physics in Great Britain to-day is inferior to what it was when Kelvin flourished, while both pure mathematics and experimental physics receive much more attention than ever before. It is probable that mathematical physics has its potential British supporters enticed away on one side by the splendid successes of experimental physics, or on the other side by the new vigour and enthusiasm that has reanimated pure mathematics. Fifty years ago pure mathematics, in the modern sense of the term, was almost unknown in this country. We had become cut off from the main current of thought and had drifted into a backwater. Instead of investigating matters of real scientific interest, men wasted their energies in concocting artificial examination problems of ever-increasing complexity. There was a real danger that mathematical progress would be choked by the Mathematical Tripos. But a few enthusiasts, inspired by the great French treatises on analysis, have now completely transformed the situation. By their lectures and their books, which have a deptn and thoroughness hitherto found only in continental treatises, they have captured the imagination of a large proportion of the most able students, who are encouraged to engage in researches on such subjects as integral equations, divergent series, or the theory of numbers. As for experimental physics, this has never been more flourishing. The far-reaching investigations into the constitution of the atom, which were originally largely due to Sir Joseph Thomson, who commenced his career as a mathematical physicist, have now, under Sir Ernest Rutherford, Sir William Bragg, Dr. Aston and others, developed on purely experimental lines, on which they have obtained results of fundamental importance. As a natural result,

young physicists devote themselves to the study of these experimental methods and pay little attention to anything involving mathematical reasoning.

There is at the present time a regrettable tendency for mathematics and physics to lose the close connection with each other that formerly existed. No doubt in Victorian days the connection was rather oppressive on one side. It is said that Kelvin asked a student to define a differential coefficient. The student commenced, " It is the limit ... ," but was cut short with the impatient rejoinder, " Wrong ! It is a velocity." Bacon's estimate of the position of mathematics found much support. " Mathematical science, he says, is the handmaid of natural philosophy ; she ought to demean herself as such ; and he declares that he cannot conceive by what ill chance it has happened that she presumes to claim precedence over her mistress " (Macaulay's " Essay on Lord Bacon "). However, the times have now changed. Pure mathematics has asserted its claims to liberty and equality, but is inclined to forget the advantages of fraternity. It is certainly true that the subject is worthy of attention for its own sake, independent of all applications, and that it is sometimes desirable to take great pains to deduce one theorem from another, although the second is apparently as obvious as the first. Elaborate investigations are often required to determine exactly the necessary and sufficient conditions for an apparently obvious theorem to be true. In work of this kind a knowledge of physics is of no advantage, and, in fact, may even be a handicap, for the intuition upon which all the greatest physicists have relied so much is a very deceptive guide in the abstract realms of the theory of functions. These considerations lead young mathematicians to neglect physics, just as the young physicists are neglecting mathematics. Both seem to be in fault in this matter. In the past, pure mathematics and physics were always helping each other by furnishing suggestions for each other's development. At the present time this mutual help is not very noticeable, but this may be only a passing phase. The mathematician who ignores physics may be ignoring ideas that would be valuable in his own subject. The physicist who devotes himself solely to experimental work may, if he ignores the theories of mathematical physicists, be ignorant of the direction in which experimental investigations of a new kind are likely to prove most successful. In addition to this disadvantage, it is very bad from an educational point of view to learn only the results formulated by Einstein, Planck, Bohr and Sommerfeld without attempting to follow their reasoning. It must be admitted that some of this reasoning involves very difficult mathematics, but other portions are easy, and these should be mastered, unless physics is to become a matter of faith rather than of reason. Physicists often laugh over the pronouncement of Todhunter concerning the place of experiments in the study of mechanics. " If a man," he wrote, " does not believe the statements of his tutor—probably a clergyman of mature years, recognised ability and blameless character —his suspicion is irrational, and manifests a want of the power of appreciating evidence, a want fatal to his success in that branch of science which he is supposed to be cultivating." Mathematicians may in their turn laugh (unless they prefer to weep) when they find physicists accepting results purely on authority, without even the hall-mark of ordination. Perhaps, however, this is merely due to humility. The experimenter is inclined to regard Einstein with the sort of reverence shown by the old Scotswoman who, when asked if she understood the pronouncements of her minister, replied, " I wudna hae the presumption."

The tendencies of modern pure mathematics and experimental physics furnish a partial explanation of the unsatisfactory position of mathematical physics in this country. Yet the explanation is not complete, for mathematical physics is flourishing enough on the continent, particularly in Germany. The suspicion arises that there is something in the British treatment of the subject which is in need of reform. One weak spot is very obvious : our

courses are overloaded with obsolete or trivial matters which have accumulated in the long years of examiners' efforts to find something new and difficult for the problem paper. The treatment of projectiles in books on elementary dynamics is a glaring example. Even in the old days, when muzzle velocities were so low that it was reasonable to neglect air resistance to a first approximation, the problems were remote from reality and were often merely disguised exercises on the properties of the parabola. At the present time, when muzzle velocities have increased so greatly, it is ridiculous and misleading to lay such stress upon problems of the old type, which usually do not trouble to mention that the resistance of the air has been neglected. However, advanced books on dynamics contain something even more ridiculous, namely, problems in which the resistance of the air is to be assumed proportional to the weight of the projectile but independent of its velocity! At least one university sets questions of this type fairly regularly. Still we must not condemn books on dynamics too severely, in spite of their rather excessive fondness for fantastic laws of force. Many parts of the subject are capable of experimental verification, especially those which relate to vibrations. The discussion of gyroscopic motion can be made very convincing with suitable apparatus, and the surprising nature of the results never fails to excite deep interest.

The latest school text-books on statics are much more physical than the older ones. We are now getting away from the six simple mechanical powers, including the frictionless wedge, the three kinds of levers, and the three systems of pullies. These have a mediaeval flavour, like the seven deadly sins, and we endeavour to avoid them. Unfortunately statics is still encumbered with problems which are chiefly rather tedious trigonometry. University courses in statics contain some interesting matter concerning chains, but as usual the discussion is over-elaborated and swollen by the introduction of such topics as chains under the action of central forces. The parts concerned with systems of forces in three dimensions contain a great deal of heavy analysis that seems to lead nowhere.

We now turn to hydrostatics. The fundamental difficulty in writing a book on this subject is that the really important theorems and problems are few. To make a book of the ordinary size it is necessary to pad it out with fantastic laws of density or other irrelevant matter. But we must beware of judging books on hydrostatics uncharitably. It is quite possible that matter which appears to have been introduced just to fill out the book has in fact arisen quite naturally in response to the demands of some forgotten examiner, and has kept its place by mere inertia.

In the case of hydrodynamics there is no lack of material. The subject is an enormous one, and it is still growing. Some parts of it, such as wave-motion, are of physical interest, and Prof. G. I. Taylor's investigations, which combine mathematics and experiment, are bringing to light real properties of real fluids. Yet it is not unfair to say of the subject as a whole, in its present state, that it is essentially unreal. It is to a great extent made up of long and difficult analytical investigations leading to results which are grotesquely different from those observed experimentally.

The other parts of applied mathematics call for less comment. The treatment of electricity and magnetism is on the whole satisfactory, though too much attention is paid to electrostatics and too little to electrodynamics. Physical optics should have much of the time at present devoted to geometrical optics, which is largely an artificial subject. Thermo-dynamics is often ignored. The quantum theory of spectral lines is a new subject that is being introduced; there are other branches of quantum theory, but these are more in the background. Relativity is not often studied seriously, in spite of the wide-spread sale of expositions of the theory for the general reader. Probably the difficulties of tensor calculus deter a great many who are interested.

Is it not possible to eliminate the obsolete or fantastic portions of the older subjects and devote the time and energy saved to the leading features of the newer ones? Of course, a merely mathematical treatment of electricity or physical optics is inadequate. It is necessary to witness experiments. On the other hand, it is not absolutely necessary for those who intend to advance mathematical physics to do experiments themselves, though this is an advantage if time allows. But with so much to learn it is unreasonable to expect the same man to be expert on both sides. The mathematical physicist must be able to appreciate the experiments performed by others. His own task is to work out a theory to embrace all the phenomena discovered by experimenters. This task is a colossal one. Kelvin nearly accomplished it, as far as the phenomena known in his time were concerned, but in the end he was baffled. Since that time new facts have accumulated and made the task still more difficult. At present some facts are explained on one theory and others on a contradictory theory. The task of discovering a wider theory that will cover all the facts and include dynamics, electricity, optics, relativity and quantum theory is one that will need many workers. Surely the countrymen of Newton, Green, Hamilton, MacCullagh, Stokes, Kelvin, Maxwell and Rayleigh will be prominent in the struggle. H. T. H. Piaggio.

University College, Nottingham.

GLEANINGS FAR AND NEAR.

258. In my Cambridge days . . . Lunn's brilliant mathematics won him a high degree [4th Wrangler, 1853] and the usual fellowship, but his oddities made the world shy of him, and he lived and died almost unknown to fame. . . . The first time I breakfasted with him at St. John's, he plunged four eggs into a saucepan filled with boiling water, then flew to the piano, and after dashing through the Overture to *Figaro*, which took two minutes and a half, proclaimed that the eggs were ready. In his early days, Lunn had attended a rehearsal of *Elijah* under Mendelssohn's own direction. He was absorbed at that time in abstruse mathematical problems, and had such a power of mental abstraction, that in " an island full of strange noises," he could read for college examinations undisturbed by outward sounds; but on this occasion he wrote out from memory the quartet " O come, every one that thirsteth," which had taken his fancy, and the copy contained but one wrong note.—Arthur Duke Coleridge, *Reminiscences* (1921), pp. 114-115.

259. " It is marvellous how like a boy, say up to twelve or thirteen, from the Solomon Islands is to a boy from Belgravia. In point of adaptability to circumstances, I should be inclined to give the palm to the former, but *quâ* pickle and jokes, etc. etc., all that constitute small boy nature, even to tears in their trousers on all occasions, etc. etc., I don't think there is a pin to choose. Darwin and Co. may say what they like, but my fellows who can't take four from five are not at all different from two of my greatest friends at Eton and Cambridge, one of whom was asked what a stalactite would melt in three hours if it melted an inch in two, and fled at the bare word; and the other learnt his Euclid by heart, signs and all, from sheer inability to comprehend it. I say it is all nonsense to say that these fellows are not capable of higher training because they are dull at first, or to compare them with those who have had all the weight of thousands of years of at least partial civilisation to start with, and whose common everyday facts would be great discoveries to these fellows."—From a letter by Bishop Selwyn.

260. Adam Riese—the Cocker of Germany—uses +, −, and is said to have introduced $\surd$ in his Rechnung . . . 1605.—*v.* Sotheran, *Bibliotheca Chem. Math.*, i. 713.

NOTE ON THE PARALLEL-POSTULATE

By A. A. Krishnaswami Ayyangar, M.A.; L.T.

Some teachers may not have noticed that there is a real connection between the parallel postulate and another and more fundamental property of space which it is also more natural to assume, *viz.* that stated by John Wallis in the seventeenth century:

"To every figure there exists a similar figure of arbitrary magnitude."

One can, in fact, be deduced from the other by rigid geometrical reasoning.

Prof. M. J. M. Hill and Prof. Nunn have developed the parallel postulate from the similarity postulate in the columns of the *Mathematical Gazette*, Dec. 1923 and May 1922. The method which I sketch below is * different from theirs and appears to be comparatively easy from the pupils' point of view.

I. Let us assume in place of Playfair's Axiom a modified form of *Euclid* VI. 4, *viz.*:

1. If two triangles have two angles of the one equal to two angles of the other, then the triangles are similar, *i.e.* the remaining angles are equal and the corresponding sides proportional.

We shall call this "the particular postulate of similarity" to distinguish it from Wallis's "General Postulate."

From this assumption, we can first deduce the other sets of necessary and sufficient conditions for the similarity of two triangles.

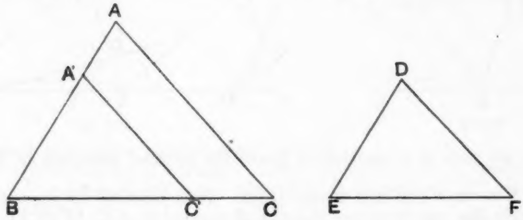


Fig. 1.

2. Thus, (*vide* Fig. 1.) if two triangles ABC, DEF have an angle B of the one equal to an angle E of the other and the sides about these proportional and if $AB > DE$, set off on BA, $BA' = ED$ and at A' make $B\hat{A}'C' = B\hat{A}C$. Then $A'C'$ cannot cut AC and therefore must cross the boundary BC at some point, say C'. By the particular postulate of similarity enunciated above, the triangles ABC, $A'BC'$ are similar and therefore

$$\frac{BC}{BC'} = \frac{BA}{BA'} = \frac{BA}{ED} = \frac{BC}{EF}$$

$$i.e.\ BC' = EF.$$

It follows easily from the above that $\triangle A'BC' \equiv \triangle DEF$ and hence the triangles ABC, DEF are similar.

* In a footnote on p. 167, vol. xii., *Mathematical Gazette*, July 1924, Prof. M. J. M. Hill remarks that either his proof or Prof. Nunn's or some equivalent proof should find a place in every text-book of elementary geometry in place of Euclid's Postulate of Parallels. I have been therefore tempted to sketch here an equivalent proof, which, I believe, will be at least as commendable as the others. On p. 413, vol. xi., Prof. Hill mentions that he has received a full statement of Prof. Nunn's Arguments which, so far as I know, has not been published anywhere, *in extenso* and I do not know therefore how far I may have been anticipated by him.

3. In the same way, we can prove the remaining sets of necessary and sufficient conditions for the similarity of triangles.

4. Next, we take up the angle-sum property of a triangle.

Let ABC be any triangle; join D, E, F the middle points of the sides BC, CA, AB. (*Vide* Fig. 2.)

From the conditions of similarity proved above the five triangles ABC, AFE, FBD, EDC, DEF are similar.

Denoting the common angle-sum of each triangle by x, we get the sum of all the angles of the last four triangles

$$\begin{aligned} &= 4x \\ &= \text{the sum of the angles of the triangle } ABC \\ &\qquad + \text{the sum of the angles at } D, E \text{ and } F, \\ &= x + 3\pi; \end{aligned}$$

$$\therefore\ x = \pi.$$

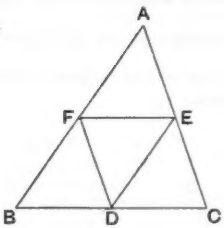


FIG. 2.

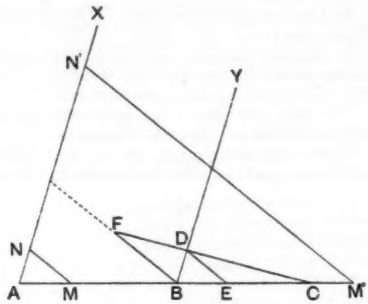


FIG. 3.

5. We are now in a position to prove the parallel postulate in Playfair's form.

Let AX be any line and B any point. It is required to show that only one straight line can be drawn through B parallel to AX. (*Vide* Fig. 3.)

Join AB and produce it to C making $BC \leqslant AB$ * and at B in BC make $Y\hat{B}C = X\hat{A}C$. Then we know $BY \parallel AX$. Let BF be any line within the angle ABY. Since F, C are on opposite sides of BY, FC must cut BY at some point D. Since the exterior $B\hat{D}C >$ the interior opposite $F\hat{B}D$, make $B\hat{D}E = F\hat{B}D$ and since DE falls within the angle BDC, it must cut BC, say at E.

Now $$A\hat{B}F + F\hat{B}D + E\hat{B}D = \pi$$
$$= B\hat{D}E + D\hat{E}B + E\hat{B}D$$
(from the angle-sum property);

$$\therefore\ A\hat{B}F = D\hat{E}B$$

Since AX, BF make the same angles with AB that BD and ED make with BE, AX, BF meet each other by the principle of similarity.

In the same way, it may be shown, by producing XA and YB beyond A and B respectively and repeating the above constructions on the other side of AB that any straight line through B within the angle YBC also cuts AX.

---

* Vide ii. (3) infra where we set off $AM = BE$ in AB and in order that M may fall between A and B, AB should be greater than BE and this will be secured if $AB \geqslant BC$, BC being obviously greater than BE.

In other words BY is the only straight line through B parallel to AX, *i.e.* two straight lines through B or two intersecting lines cannot both be parallel to another straight line.

II. We may even consider the constant angle-sum property of a triangle as a fundamental axiom (though according to Prof. Nunn, one may miss the philosophical view of geometry) and thereby bring out more clearly the arbitrary nature of our assumptions, which seem plausible in the other cases.

Assuming then that the sum of the three angles of a triangle is constant, let us build up the theory of parallels.

1. Let ABC be any triangle; join A to any point D in BC. (*Vide* Fig. 4.) If the constant angle-sum be α, we form the equation

$$2\alpha = \text{the sum of the angles of the triangles } ABD \text{ and } ADC$$
$$= \text{the sum of the angles of the triangle } ABC + \text{the sum of the angles at } D.$$
$$= \alpha + \pi$$
$$\therefore \ \alpha = \pi$$

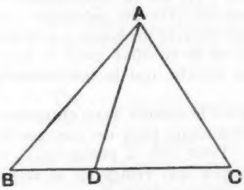


FIG. 4.

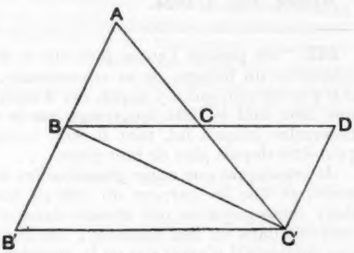


FIG. 5.

2. We shall next prove the following theorem:

If the two sides AB, AC of a triangle ABC be produced to B', C' respectively, such that $BB'=AB$ and $CC'=AC$, then $B'C'$ is parallel to BC and equal to $2BC$. (*Vide* Fig. 5.)

Produce BC to D making $CD=BC$; join DC'; then clearly $\triangle ABC \equiv \triangle C'DC$ so that

$$\left.\begin{array}{l} A\hat{B}C = C'\hat{D}C \\ AB = C'D \quad i.e. \ BB' = C'D \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If BC' be joined,

$$B'\hat{B}C' = B\hat{A}C' + B\hat{C'}A$$
$$= A\hat{C'}D + B\hat{C'}A$$
$$= BC'D \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Hence, from (i) and (ii)

$$\triangle B'BC' \equiv \triangle DC'B, \quad \text{showing that}$$
$$B\hat{B'}C' = C'\hat{D}B = A\hat{B}C \quad \text{by (i)}$$

and $$B'C' = BD = 2BC.$$

Thus $B'C'$ is parallel to BC and equal to $2BC$.

***Cor.* If AB, AC be produced to M, N respectively such that $AM = 2^n . AB$ and $AN = 2^n . AC$ where n is a positive integer, then MN will be parallel to BC and equal to $2^n . BC$.**

3. Lastly, to prove the parallel postulate (*vide* Fig. 3). Proceeding with the same construction as in I. 5 above we can find a triangle BDE such that BD is parallel to AX and ED to BF. On AB and AB produced we can find points M, M' such that $AM = BE$ and $AM' = 2^n BE > AB$ (n being a positive

integer suitably chosen so as to make $2^n . BE > AB$). On AX, take N, N' such that $AN = BD$ and $AN' = 2^n BD$.

By the corollary above, MN and $M'N'$ are parallel to each other. Since BF also makes the same angle with AB that MN, ED, $M'N'$ make with it, BF is also parallel to MN and $M'N'$ and must cross the boundary of the quadrilateral $MNN'M'$ at some point in NN', *i.e.* BF must cut AX.

Hence the parallel postulate as before.

In conclusion, I wish to remark, that in actual class-teaching we may substitute for Playfair's Axiom *Euclid* I., 29 or 32 (which can be roughly verified by drawing and measurement) and deduce other properties of triangles and parallels therefrom. Firstly, such a procedure diminishes by one the number of theorems which pupils have to study and, secondly, it emphasizes the empirical character of the axiom by giving to it a form which challenges attention. It may be well to point out finally that the assumption of the angle-sum property of a triangle as an empirical axiom has the further advantage—that it relates to bounded space.

Mysore, Aug. 5, 1924. A. A. KRISHNASWAMI AYYANGAR.

---

261. " Je passois l'autre jour sur le Pont-Neuf avec un de mes amis : il rencontra un homme de sa connoisance, qu'il me dit être un géomètre ; et il n'y avoit rien qui n'y parût, car il étoit dans une rêverie profonde : il fallut que mon ami le tirât longtemps par la manche, et le secouât pour le faire descendre jusqu'à lui, tant il étoit occupé d'une courbe qui le tourmentoit peut-être depuis plus de huit jours. . . .

Je remarquai que notre géomètre fut reçu de tout le monde avec empressement, et que les garçons du café en faisoient beaucoup plus de cas que de deux mousquetaires qui étoient dans un coin. Pour lui, il parut qu'il se trouvoit dans un lieu agréable ; car il dérida un peu son visage, et se mit à rire comme s'il n'avoit pas eu la moindre teinture de géométrie.

Cependant son esprit régulier toisoit tout ce qui se disoit dans la conversation. . . . Un nouvelliste parla du bombardement du château de Fontarabie ; et il nous donna soudain les propriétés de la ligne que les bombes avoient décrite en l'air ; et, charmé de savoir cela, il voulut en ignorer entièrement le succès. . . .

Un moment après il sortit, et nous le suivîmes. Comme il alloit assez vite, et qu'il négligeoit de regarder devant lui, il fut rencontré directement par un autre homme : ils se choquèrent rudement ; et de ce coup ils rejaillirent chacun de leur côté, en raison rèciproque de leur vitesse et de leurs masses. . . ." —Montesquieu : *Lettres Persanes*, lettre CXXVIII. (*Oeuvres*, 1822, t. vi. p. 335). [*Per* F. Puryer White, M.A.]

262. *Gorham v. the Bishop of Exeter* arose from a difference of opinion as to the fitness of an accomplished mathematician, the Rev. George Cornelius Gorham, to be the incumbent of a Crown living. [Gorham was 3rd Wrangler and 2nd Smith's Prizeman, 1808, and Fellow of Queen's.]

263. I haven't forgot fractions and logareems, and practice, and so on to algebrae, where it always seems to me to blow hard, for whizz goes my head in a jiffy as soon as I've mounted the ladder to look into that country. . . . —(Farmer Fleming *loq.*) *Rhoda Fleming*, 1890, p. 19.

264. It is curious to compare eighteenth-century complaints of the University's meanness touching mathematical types—" they used daggers turned sideways for *plus's* "—with a tribute paid to the Cambridge University Press in 1918, " which in setting up over five hundred quarto pages of numerical tables has allowed less than a dozen printer's errors to pass its proof-readers, and has, in addition, frequently queried our own mistakes."—Professor E. W. Brown. *Tables for the Motions of the Moon.*

PARALLELISM AND SIMILARITY.

By D. K. Picken, M.A.

1. The recent discussion of this subject * cannot fail to have important effects. It must certainly have aroused in many a new conception of the part Similarity should play in Elementary Geometry.

In this article—which embodies an integral part of work on elementary geometry, most of which has recently been published †—a presentation of the ideas of Parallelism is given, which is designed to bring the conviction on this subject that has too commonly been lacking, and to show how central its place is in Euclidean Geometry. Together with the discussion of the Plane theory, an indication is given (in § 5) of a sufficient elementary treatment of Similarity: "sufficient," in no disparaging sense, but in the sense of sufficing for every reasonable need at the stage in question.

But it must be emphasised that, since any complete theoretical treatment of Similarity necessarily involves the general problem of Ratio—which is essentially abstruse—there can, in the last analysis, be no theoretical priority of Similarity over Parallelism—for which *all* the facts are relatively simple. It is important to observe that such phrases as "figures similar to a given figure upon *every possible scale of magnitude*" ("Report," p. 38) have implicit in them the difficulties inherent in the concept of Ratio‡.

Parallel Lines.

2. (i) The central proposition in this connection is Euc. I. 16, which may be restated thus:

If A, B, C, ... are points on order on a straight line X′X, and P a point not on that line, the "concave" angles PAX, PBX, ... are in order of magnitude—being *increasing* order of *absolute* magnitude§.

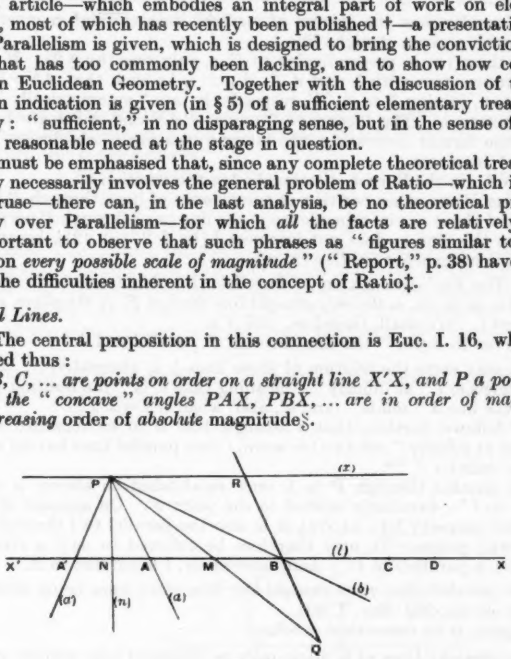


Fig. 1.

The Euc. proof of this proposition—in which we shall suppose the line PQ used, such that the mid-point M of AB is also the mid-point of PQ—depends directly on fundamental principles of Euclidean Geometry.

* *Gazette*, May 1922, Dec. 1923, etc.; *Report on the Teaching of Geometry in Schools* (referred to hereafter as *Report*, pp. 35-40.

† *Gazette*, Dec. 1922; *Proc. of London Math. Soc.* ii. 23, part i. p. 45.

‡ Articles on "Ratio and Proportion," by the writer, in the *Gazette*, Jan. and May 1920, may be referred to here—in order to save the space that would be occupied by restating what is set out at length there. On p. 10 of that volume (vol. x.) the relation between the concepts of Ratio and of the Real Numbers is shown. This I have elaborated in a little book on Number recently published.

§ Observe that the statement covers the other case, of the concave angles PAX', PBX', ... which are in *decreasing* order of absolute magnitude—since $\angle XAP + \angle PAX' =$ st. $\angle$.

(ii) (1) An immediate corollary may be stated as follows:

If intersecting straight lines l_1, l_2 both intersect another straight line l, then

$$(l, l_1) \not\equiv (l, l_2)^*.$$

Whence, again, conversely,

(2) *If straight lines l_1, l_2 both intersect another straight line l, and if*

$$(l, l_1) \equiv (l, l_2)$$

—all three lines being in one plane—*the lines l_1, l_2 do not intersect one another.*

3. (i) Consider now the system of straight lines through a given point P coplanar with a given straight line l.

One of these (n) is perpendicular to l† ; only one [by § 2]. If N be its point of intersection with l, and if A, A', on l, be such that $AN = NA'$, then the lines a, a', joining P to A, A', respectively, are "symmetrically" placed ‡ with respect to n. The "concave" angles PAN and NPA are both acute [by § 2], the former decreasing, the latter increasing ("absolutely"), as the length NA increases.

When $NA = 0$, the lines a, a' coincide—in the position n.

And there is one other position of coincidence of lines through P which are symmetrical about n, viz. coincidence in the line (m) § through P perpendicular to n. But this line m does not intersect l [by § 2, (ii), (2)], because of the Right Angle congruence $(n, l) \equiv (n, m)$||.

(ii) (1) The Euclidean Axiom of Parallels (Playfair form) is the proposition that *the line m*, in (i), *is the only straight line through P, in the plane, which does not intersect l.* We shall, therefore, call it x.

This serves to *define* "the line through P *parallel* to l."

(2) We may state the relation of these lines l, x, alternatively, in the form that x intersects l "at infinity"—meaning that it is "the limiting position" to which the line a "tends" (and a', also) when $NA \to \infty$.

And it follows, further, that a straight line is to be regarded as having *one* "point at infinity," not two (or more): two parallel lines having a common "point at infinity." **

(3) The parallel through P to l, once established as above, is not—as a "parallel to l"—peculiarly related to the point P. On account of the non-intersection property [(1), above], it is also the parallel to l through any one of its (own) points. It may therefore be referred to as "a straight line parallel (or, a parallel) to l"; and, conversely, l is parallel to it.

(4) *Two parallels to a given straight line* (the three lines being coplanar) *are parallel to one another* [Euc. I. 30].

This, again, is an immediate corollary.

(5) The straight lines of a plane may be classified into infinite systems of parallels—all the infinity of parallels of one such system having one common "point at infinity."

(iii) Reverting to the facts of § 2, (ii), we see that

straight lines which form, with a transversal, congruent Complete Angles are parallel [Euc. I. 27-8]; and, conversely,

---

* See *Gazette*, vol. xi. p. 188 (Dec. 1922).

† See Carslaw's *Non-Euc. Geom. and Trig.* (Longmans), pp. 6-8, on the basic constructions of Plane Geometry.

‡ Cf. *Report*, p. 48. Symmetry has great value in concise expression.

§ Afterwards called x, and so in Fig. 1.

|| See *Gazette*, vol. xi. p. 189.

** It is to be clearly understood that all references to "infinity" are simply abbreviated "limit" statements, which require interpretation in terms of the more elementary ideas of the context.

two parallel lines form, with ANY *transversal, congruent Complete Angles* * [Euc. I. 29].

(The existence, and the uniqueness, of the non-intersecting straight line is the reason for the simplicity of the argument.)

The angle-sum.

4. (i) (1) The proposition [§ 2, (i)] of Euc. XI. 16 is completed—as in Euc. XI. 32—by means of the fact that the line BQ is parallel to AP.

If BR be the extension of QB—so that R is to the same side of the line l as P is †—we have

$$\text{concave } \angle\text{s } XBR,\ RBP,\ PBA = BAP,\ APB,\ PBA.$$

Hence the sum of these latter $\angle$s $= \pm$st. $\angle$.

Thus, for triangles (ABC), in general,

$$|\angle A + \angle B + \angle C| = \text{st. } \angle.$$

(2) An immediate corollary, *affected by sign*, is as follows:

If D be on either extension of BC (*i.e.* collinear with B, C but not "between" them †), then (for *concave* angles)

$$\angle BAC = \angle ABD - \angle ACD (= \angle DCA - \angle DBA).$$

(ii) Using (i) with the facts of § 3, we see that

$$\angle PAN \to 0 \quad \text{when} \quad A \to \infty,$$

and, therefore, that *parallel lines may be regarded as having zero-inclination.*

This, taken in conjunction with § 3, (iii), is the basis of *the conception of "Direction"* in Geometry—in accordance with which [see appended Note on Direction] a plane has belonging to it a system of directions, specifiable by means of all the straight lines, in it, through *any* one of its points.

Similarity.

5. The essentials of an elementary treatment of Similarity may be demonstrated from a single figure, as follows:

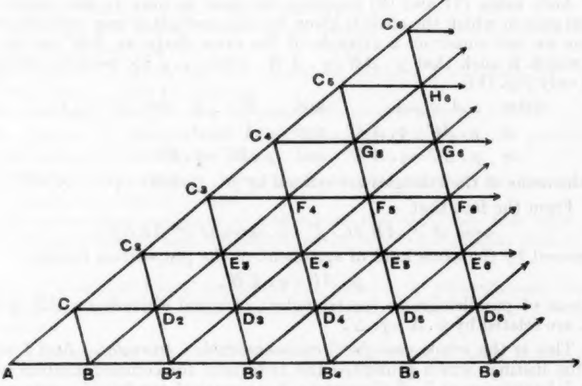


FIG. 2.

* In the article in the *Gazette*, vol. xi. (p. 190), the theory of Parallels was taken up at this point.

† Here we come expressly upon the "axioms of order," which have generally been left (unformulated) to the learner's intuitions in Elementary Geometry. For these see Whitehead's *Axioms of Descriptive Geometry* (Cambridge Tracts, No. 5), ch. i., especially p. 8.

(i) Setting out from any triangle ABC, extend AB, AC indefinitely. Parallels through B and C to AC and AB, respectively, intersect at the point D_2; and the parallel through D_2 to BC intersects the lines AB, AC in B_2, C_2. Parallels through B_2 and C_2 to AC and AB, respectively, intersect the previous parallels CD_2 and BD_2 at D_3 and E_3, respectively; and intersect one another at E_4. The line D_3E_3 (which is parallel to BC) intersects the lines AB, AC at B_3, C_3. And so on indefinitely. (Note that, in the scheme of notation, the line B_nC_n is parallel to BC and contains $D_n, E_n, \ldots L_n$—where the number of symbols $D, E, \ldots L$ is $n-1$; each parallel to AB is a line of K's, where K is first D, then E, and so on; each parallel to AC is a line $D_mE_{m+1}F_{m+2}\ldots$; and B, C are also B_1, C_1.)

We have thus a network of triangles each of which is either congruent or contra-congruent with the original triangle. And from it we have immediately the following facts:

(1) The triangle AB_nC_n is such that

$$\angle B_n = \angle B, \quad \angle C_n = \angle C; \quad AB_n = n \,.\, AB, \quad AC_n = n \,.\, AC, \quad B_nC_n = n \,.\, BC;$$
$$\text{and} \quad \triangle AB_nC_n = n^2 \,.\, \triangle ABC.$$

It is therefore a triangle "of the same shape*" as ABC, on the n-multiple "scale."

And, using the facts of Triangle Congruence, it follows that we can construct a triangle of the same shape as ABC on *any* line $A'B'$, $= n \,.\, AB$, using only

either $\angle A' = \angle A$ and $\angle B' = \angle B$ (or $\angle C' = \angle C$),
or $A'C' = n \,.\, AC$ and $\angle A' = \angle A$,
or $A'C' = n \,.\, AC$ and $B'C' = n \,.\, BC$.

(2) The construction for dividing a given line into n equal parts—or, for finding the n^{th} part of a given length—is an immediate corollary.

(3) Using (1) conversely, in conjunction with (2), we have the proposition corresponding to (1) for the *n-sub* multiple scale.

(4) And, using (1) and (3) together, we pass at once to the much more general case in which the scale is given by *any multiple of any submultiple.*

Thus we can construct a triangle of the same shape as ABC on any line $A'B'$ which is such that $p \,.\, AB = q \,.\, A'B'$, where p, q are positive integral—using only [cp. (1)]

either $\angle A' = \angle A$ and $\angle B' = \angle B$ (or $\angle C' = \angle C$),
or $p \,.\, AC = q \,.\, A'C'$ and $\angle A' = \angle A$,
or $p \,.\, AC = q \,.\, A'C'$ and $p \,.\, BC = q \,.\, B'C'$.

And the areas of the triangles are related by $p^2 \,.\, \triangle ABC = q^2 \,.\, \triangle A'B'C'$.

(5) From the fact that

$$\text{area of } \|^{\text{m}}\ AB_nD_{n+1}C = n \,.\, (\text{area of } \|^{\text{m}}\ ABD_2C),$$

we proceed by the same kind of argument to the proposition that if

$$p \,.\, AB = q \,.\, A'B',$$

the areas of parallelograms (or triangles), of equal altitude on AB, $A'B'$ as bases, are related by $p \,.\, \Delta = q \,.\, \Delta'$.

(ii) This is the whole case *for* "*commensurable*" *triangles.* And I submit that the instinct which prompted the treatment for commensurables only—as an "improvement" of *elementary* "geometrical teaching"—was a right instinct. It has, however, never been logically followed out†, because of the temptation to state propositions in the more general form, whenever (as

* "Sameness of shape"—or "similarity"—being defined in this connection, first for an integral "scale."

† Nor are the elementary ideas of Ratio satisfactorily taught. See *Gazette*, vol. x. p. 9.

commonly) there is no apparent geometrical separation between the cases of commensurables and of incommensurables.

It seems to be important that beginners should be taught: (1) that there is a highly important *theoretical* distinction between commensurables and incommensurables; but (2) that no *practical* process of measurement can ever separate the types—and therefore that the commensurable case is *practically* sufficient; (3) that all the standard propositions remain true in the general case, but that the *proof* of this generalisation raises essential *theoretical* difficulties, as to Ratio and Number, which cannot be resolved at an early stage of a mathematical course.

The more effectively learners can be prevented from ignoring the existence of the essential theoretical difficulties, the better they will be prepared for facing them later on*.

Parallelism in Three Dimensional Geometry.

6. (i) We set out from the axiomatic propositions†—

(1) that the straight line through two points of a plane is wholly in the plane;

(2) that there is *one* plane through any three given, non-collinear points—

of which (2) can be immediately expanded in the forms: one plane (*a*) through given straight line and non-collinear point, or (*b*) through two given intersecting straight lines, or (*c*) through two given parallel lines‡.

(ii) For relationship of Straight Line and Plane, in Space, there are four logical possibilities:

(1) the line may be in the plane;

(2) the line may be parallel to a straight line in the plane, while not itself in the plane—which may easily be seen to imply non-intersection [by (i)];

(3) the line and plane may have one point in common;

(These three are clearly all actual types.)

(4) the line *might*, conceivably, neither intersect the plane nor be parallel to any line in it.

The relation (2) gives the best means of defining ***Parallelism for Straight Line and Plane.*** [See § 7, below.]

The extension of (4) is then an axiomatic proposition, as follows:

(iii) Axiom. ***A straight line, which is neither in a plane nor parallel to it*** (as just defined), ***intersects the plane.***

(iv) (1) It is immediately deducible from (iii)—taken in conjunction with the "axioms of order§"—that

if two planes have one point of intersection, they must also have other points of intersection‖—and, therefore [by (i), above] *a straight line of intersection.*

Cor. If a straight line, l, is parallel to a plane α, the plane through l and any point of α intersects α in a parallel to l. (A consequence of the non-intersection properties.)

(2) There remains the possibility that *all* the straight lines in one of two

* The mode of transition to the general case is indicated in the *Gazette*, vol. x. p. 10.

† The aim should be to depart as little as possible from standard elementary formulations of axioms (in so far as these are sound) while taking the utmost pains to keep in line with modern ideas—as presented in Whitehead's tracts.

‡ The relation of (*c*) to the facts of § 3, above, should be noted.

§ See footnote to § 4 (1), (2), p. 197.

‖ To take this fact as the axiom—which is implicit in Euclid, explicit in Hilbert—appears to me objectionable: the more so when it is stated in the form "at least one other."

planes may be parallel to the other plane—in which case the two planes would have no point of intersection.

To this case we return in (vii) below.

(v) (1) We are now in a position to prove the Space proposition [cp. § 3 (ii), 4] that

*parallels to a given straight line are parallel to one another**.

Given that l_1, l_2 are both parallel to l, denote the planes of l, l_1 and of l, l_2 (supposed non-coincident) by α_1 and α_2—so that line l_1 is parallel to plane α_2, and line l_2 to plane α_1.

The plane through l_1 and any point, P_2, of l_2 intersects plane α_2 in the line through P_2 parallel to l_1—say l_2'; and l_2', which is also coplanar with l, cannot therefore intersect l†; it must, therefore, be the line through P_2 parallel to l. Hence l_2 coincides with l_2' and is parallel to l_1.

[*Note.*—This proposition exalts Parallelism of Straight Lines from merely a Plane property to a Space property, and is of the utmost importance to the conception of Direction.]

(2) From (1) we have [as in Euc. XI. 10] the proposition that

if two intersecting straight lines are respectively parallel to two other intersecting straight lines, then the two Complete Angles are congruent.

(This also is important to the conception of Direction.)

(vi) *The sub-system of directions which belongs to a plane*‡ *is characterised by the fact that all of these directions are at right angles*§ *to one pair of opposite directions.*

This important proposition is proved in four successive steps:

(1) *If two straight lines l_1, l_2 are concurrent with a given straight line l (at point O) and are both perpendicular*§ *to it, then every straight line through O, coplanar with l_1, l_2, is perpendicular to l.* [Cp. Euc. XI. 4||.)

(2) Conversely, *every line through O perpendicular to l is coplanar with l_1, l_2.* [Cp. Euc. XI. 5.]

Def. The line l is called a "*normal*" to the plane of l_1, l_2.

(3) A *construction* can be given *for a normal to a given plane from any given point not in the plane.* [Cp. Euc. XI. 11.]

But the construction does not establish uniqueness of the normal. Hence it is necessary to prove further—

(4) that *there is only one normal from an external point.* (Proof by *reductio ad absurdum.*)

(5) The proposition as stated at the beginning of this sub-section (vi) then follows at once from the facts of (v)—the normal directions being determinate when any two (non-opposite) directions, belonging to the plane, are given. And the facts of Euc. XI. 6, 8, 12, 13 are relegated to their proper subordinate place.

The apparent complexity of directions for the Plane is thus reduced to simple expression in terms of the simple fact of directions for the Straight Line.

(vii) It follows from (vi) that two different planes are either

(1) such that there is one-one correspondence of parallelism between the straight lines in one, through any chosen point, and the straight lines in the

* The difference in mode of proof from Euc. XI. 9 is of crucial importance to the treatment. See next sub-section, (vi).

† If l, l_2' were to intersect, at a point A, both would be parallel through A to l_1.

‡ § 4, (ii).

§ It is convenient to keep the term "perpendicular" for *intersection* at right angles.

|| Note the slight difference from Euc. XI. 4, 5, in the formulation of (1) and (2).

other, through any chosen point (same normal directions, and therefore common plane sub-system of directions). See (iv), (2) above.

Or (2) they have *one* pair of opposite directions in common, viz. those of their line of intersection.

(3) The relation of (1) gives the means of defining *Parallelism of Planes*—involving, of course, non-intersection.

(4) And, further, as parallelism implies zero-inclination of normals, *inclination of planes*, in general, is *definable as inclination of normal directions*—obviously equivalent to the more intuitive conception (of greatest inclination of slopes).

[Complete Angle congruence $(n_1, n_2) \equiv (s_1, s_2)$, from Right Angle congruence $(n_1, s_1) \equiv (n_2, s_2)$.]

(5) The facts of Euc. 14-19 fall naturally into line.

7. The keynote of this discussion of Parallelism is that non-intersection is not the primary characteristic : it is a secondary characteristic, of the greatest importance to the theory (so important that it has usurped the first place in that theory). *The primary characteristic is zero-inclination*, or *common direction*—based on equality of inclination*.

The distinction would be trivial if non-intersection always implied parallelism ; but this is not, of course, the case—even among the fundamental geometrical forms discussed above. The fact that straight lines in Space may neither intersect nor be parallel has been relegated too much to the background. It is a type of fact that would assume greater prominence in Euclidean Geometry of higher dimensionality.

The fact that *parallelism* in each instance *implies non-intersection,* while *non-intersection does not necessarily imply parallelism,* is worthy of emphasis. Non-intersection is a necessary, but not a sufficient, condition of parallelism in three dimensional geometry.

Note on Direction.

The *Report* does something to give Direction—at last—its due place in Geometry, as required by commonsense and by Physical Science (Vector theory). It says (p. 42) that " when parallelism is familiar, the notion of direction is invaluable." I would go further and say that the conceptions of Parallelism and Direction must develop together—they are part and parcel of one another, like Number and Ratio.

The proper use of Direction is purely a question of careful definition. The definition must be such that

(1) a given *straight line* has *two directions*, said to be " *opposite* " ;

(2) two *intersecting straight lines* necessarily *differ in direction*†—giving two different pairs of opposite directions ;

(3) *parallel straight lines* have the *same pair of opposite directions.*

The conception of Direction appears to be peculiarly characteristic of Euclidean Space. Such a space may be said to have a system of directions belonging to it, which are specifiable by means of *all* the straight lines through *any* one of its points. Compare case of Plane [§ 4, (ii) and § 6, (vi)].

Thus in Euclidean Geometry we have Straight Line, Plane, "Space," etc., in a certain sense *directionally constant,* from point to point. This has an important relation to the fact that Euclidean Geometry is not merely one of many possible " geometries "—with more or less equal claims—but is *the basic geometry.*

D. K. Picken.

Ormond College, University of Melbourne.

* See footnote at the end.

† The difference of direction is specified by angle quantities, and parallelism is bound up with angles of inclination.

MATHEMATICAL NOTES.

727. [V. 1. a. ζ.] *On Note* 718, *Math. Gazette*, xii. 164, *The Teaching of Parallels.*

The "direction theory" of parallel lines has been exploded so frequently, and the case has been stated so well and so recently in the *Report on the Teaching of Geometry in Schools* (pp. 40-44), that some apology is necessary for taking up space on the matter. The appearance of Mr. Baxter's note, however, shows that there are probably still many teachers who have not yet realised the difficulties at issue.

He starts with the *Definition*: "The angle between two straight lines is the amount of turning each line must undergo to occupy the position of the other." If the lines intersect this conveys a picture certainly, and by means of suitable axioms of congruence it will be possible to compare angles at different points. But if the lines do not intersect it is impossible to bring them into coincidence by any rotation. His second *Definition*: "When the angle between two straight lines is zero, the lines are said to be parallel," is therefore meaningless at this stage.

Later on Mr. Baxter states a *Theorem*: "If a straight line cuts two parallel lines, the exterior angle is equal to the interior opposite angle on the same side of the cutting line," and in order to prove this theorem he states that the difference between these angles is the amount of turning, which, by hypothesis, is zero. That is, in the case of two non-intersecting lines he assumes tacitly that the amount of turning is measured by the difference between the angles which the lines make with a transversal. But in order that the measure of the angle as thus defined may be *unique*, it is necessary to prove that the difference is the same for *any* transversal. This requires a definite assumption equivalent to Euclid's or Playfair's axiom.

Whatever theory of parallel lines is chosen for elementary teaching let us at least be frank, and not slur over the difficulties in tacit assumptions. It is not logical, and teaches slovenly thinking. D. M. Y. SOMMERVILLE.

Victoria University College, Wellington, N.Z.

Mr. H. A. Baxter contributes an article to the July number of the *Mathematical Gazette* (pp. 164, 165) which cannot be allowed to pass unchallenged. The idea of rotation on which he bases his proposed treatment of angles and parallels is presumably the motion of rotation of one plane upon a fixed plane round a centre of rotation kept fixed. One can easily imagine a rotation round the axis of a frustum of a right circular cone, which would transfer one generating line of its curved surface into the position of any other generating line; similarly a rotation round the axis of a right circular cylinder would transfer one generating line of its curved surface into the position of any other generating line. I assume that this is not the kind of rotation intended.

Mr. Baxter defines an angle thus:—"The angle between two straight lines is the amount of turning each line must undergo to occupy the position of the other." Now let CA and CB be any two rays drawn from C. How can the line CA be turned in the plane ACB so as to take up a position lying along CB? It would be interesting to discuss the locus of a point O such that an appropriate rotation about O would transfer CA to lie along CB, whether the amount of turning would be the same for all suitable positions of O, and to what extent this discussion would involve any assumptions with regard to parallels. But as Mr. Baxter intends his treatment for "Boys of about 12," he presumably means that the point O should be taken in the most simple and obvious position C. I assume then that the rotation round C as centre of rotation is intended. If so, his definition of "The angle between two straight lines" involves the assumption that the two straight lines have one point in common.

His next definition, "When the angle between two straight lines is zero, the lines are said to be parallel," therefore involves the assumption that two parallel straight lines meet. Otherwise we cannot consider "the angle between them."

The next statement, "If two straight lines are drawn from a point, and the angle between them is zero, they coincide with each other," shows that parallel straight lines, as defined above, cannot be distinct from each other but must coincide. In other words, no two different straight lines can have zero angle between them.

Then Mr. Baxter treats us to a proof of the angle properties of parallels, which shows apparently once for all that a Non-Euclidean Geometry is utter nonsense. Q.E.D.

His argument, when expanded, appears to be this:—Let AB, CD be two straight lines meeting at E, and let the straight line PQR cut them. The difference of the angles RQC, RPA is the angle E, which measures the amount of turning between QC, PA. (This statement can be proved only by assuming either Euclid's parallel postulate or its equivalent.) If AB, CD are parallel, *i.e.* if AB, CD meet at zero angle, *i.e.* if AB, CD coincide, then

$$\angle RQC = \angle RPA,$$

i.e. two coincident angles are equal.

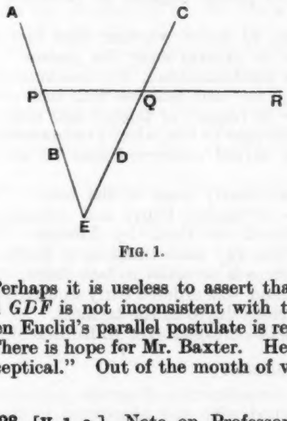


FIG. 1.

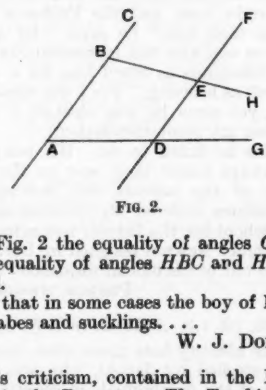


FIG. 2.

Perhaps it is useless to assert that in Fig. 2 the equality of angles GAC and GDF is not inconsistent with the inequality of angles HBC and HEF, when Euclid's parallel postulate is rejected.

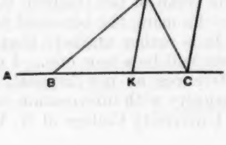
There is hope for Mr. Baxter. He finds that in some cases the boy of 12 is "sceptical." Out of the mouth of very babes and sucklings. . . .

W. J. DOBBS.

728. [V. 1. a.] Note on Professor Hill's criticism, contained in the May number of the *Gazette* (p. 78), of a point in the Report on *The Teaching of Geometry in Schools*.

The following is a portion of a proof which appears in the Report:

"ACY is an angle between ACS and ABS. There is therefore a point K between B and C such that the angle AKS is equal to the angle ACY." Professor Hill's comment on this is that this wording "will be taken to mean that as a point P goes from C to B, the angle SPA increases continuously from the value SCA to the value SBA," and he gives an illustration to show that this would not be true if the Geometry were Elliptic,



The corresponding theorem in Algebra would be: If $k > f(a)$ and $< f(b)$ the function $f(x)$ being finite, continuous, and single-valued between a and b, then there exists a value of x between a and b for which $f(x) = k$.

Surely this could not be " taken to mean " that $f(x)$ increases continuously between the values $x = a$ and $x = b$. And if not in Algebra, why in Geometry ?

Professor Hill's further criticism on the proof as making an appeal to the " Principle of Continuity " raises other and difficult points. It may well be that the proof given by him of the particular theorem in his article, " The Postulate of Parallels " (*Math. Gazette*, Dec., 1923), has an advantage in this respect. But surely his criticism that there is in the words quoted above an " obvious suggestion of continuous increase " cannot be sustained.

[I am indebted to Professor Neville for pointing out that Bolzano's investigation (1817) of the theorem in Algebra quoted above is one of the classical analyses of the obvious. It was edited by P. E. B. Jourdain for reprint in Ostwald's *Klassiker* (No. 153, 1905), and has recently been added to the Library. See p. 183 of the July number of the *Gazette*.] H. E. PIGGOTT.

729. [V. 8.] " *Mathematics enough for* your *business*."

Samuel Newton was mathematical master at Christ's Hospital from 1695 to 1708.

He was chosen from five candidates partly because he received " a good character " from Mr. Isaac Newton, and partly because he professed to " understand the Latin tounge very well."

Six months later the Committee began to doubt whether they had got the right man, and the Professor writes to explain away his praises. " I never took him," he says, " for a deep mathematician, but recommended him as one who had mathematicks enough for your business with such other quallifications as fitted him for a master in respect of temper and conduct as well as learning. For I was almost a stranger to him when I recommended him, yet since he was elected, I reckon myself concerned that he should answer my recommendation."

This he failed to do. His temper was clearly none of the best. Two runaways found their way to the office of Samuel Pepys and complained both of the labours and lashings inflicted on them by Newton. The Committee ordered Mr. Newton not to use any such thing as a fferillo in his school for the future, but rather as there is occasion to lash them.

Samuel Newton resigned in Dec. 1708 and was ordered early in Jan. 1709 to be out of his official residence " in fourteen dayes at the farthest."

PEARCE, *Annals of Christ's Hospital* (per A. Robson M.A.).

730. [C. 1.] *A Result in Differential Calculus.*

The identity here given arose from the consideration of certain polynomials first studied by Jacobi, who was concerned with one expression in simple form of the polynomials in question. Having been obliged to seek an alternative expression, I found the following to be an elementary theorem arising out of the work.

If n, r, t are positive integers,

$$\frac{d^n}{dx^n}\left\{x^{n+r}(x-1)^t\right\} = \frac{x^r t!}{(t+r)!}\frac{d^{n+r}}{dx^{n+r}}\left\{x^n(x-1)^{t+r}\right\}.$$

The truth of the theorem is easily demonstrated by expanding the powers of $(x-1)$, using the binomial theorem, and performing the differentiations.

It is rather unlikely that a result in elementary calculus obtained at this date will be a new one. I cannot, however, find mention of it in the limited references at my disposal, and shall be glad if any reader can satisfy my curiosity with information concerning any previous appearance of the identity.

University College of N. Wales. W. L. FERRAR.

731. [V. 1.] *Expanding an Allusion.*

In a review published in the May number of the *Gazette*, I have mentioned that a certain proof could be made valid though preposterous if reference was made to Abel's theorem. To avoid misunderstanding, perhaps I should have said definitely that the solidification of the proof would require not Abel's theorem itself, but one of the delicate converse theorems which, though known collectively as Tauberian, are due for the most part to Hardy and Littlewood.

E. H. NEVILLE.

May 1, 1924.

732. [X. 1.] *A Curious Method of Computation.*

The following computing method which is in use among the saw-millers in New Zealand may be of interest.

For each size of the sawn timber the numbers of pieces of each length are entered in columns, thus

	Lengths : feet.								
	24	25	26	27	28	29	30	31	32
No. of pieces	28	13	71	—	—	53	—	12	37

Then to find the total length, instead of straightforward multiplication and addition, they proceed as follows. Starting with the longest pieces the number (37) is set down. This number is added to the next $(37+12=49)$. The number (49) thus obtained is added to the next; in this case the next number is zero, and the same number (49) is written down. This process is carried out to the end. If, as in this example, there are no pieces shorter than 24 feet, the last number written down (214) is multiplied by 23. Then the column is added. It is easily seen that in the continued addition each number has been counted as often as its ordinal number in the series.

37
49
49
102
102
102
173
186
214
642
428
5936

D. M. Y. SOMMERVILLE.

Victoria University College, Wellington, N.Z.

733. [K^1. 6. a.] *The Analytical Treatment of the Epipedon.*

Certain properties of the "Epipedon," discussed by Professor E. L. Watkin in the *Mathematical Gazette* (Vol. xi. No. 167, pp. 418-420) may be obtained a alytically.

With $OPQR$ as tetrahedron of reference, let the co-ordinates of A be (a, b, c, d). Then it is easily seen that the co-ordinates of the other vertices are as follow :—

$$\begin{array}{ll} & A'(-a,\ b,\ c,\ d), \\ B(a, b, -c, d), & B'(-a,\ b,\ -c,\ d), \\ C(a, b, -c, -d), & C'(a,\ -b,\ c,\ d), \\ D(a, b, c, -d), & D'(-a,\ b,\ c,\ -d); \end{array}$$

while the equations of the faces $ABCD$, $A'B'C'D'$ are respectively

$$x/a - y/b = 0, \qquad x/a + y/b = 0.$$

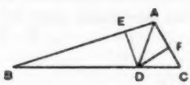
It follows that every quadric which passes through any seven of the points $A, B, C, D, A', B', C', D'$ also passes through the eighth point and has $OPQR$ or a self-conjugate tetrahedron.

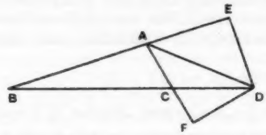
From certain results obtained by the present writer in a paper to be published by the London Mathematical Society, it may be shown that if one epipedon may be drawn so that its faces are tangent planes to one quadric and its edges are tangent lines to another quadric, then an infinite number of such epipeda may be so drawn and the two quadrics have ring-contact.

East London College.

S. L. GREEN.

734. [K^1. b. a.]. May I put forward the following proof of Euclid VI. 3, the only geometrical proportion-theorems assumed being, "The areas of triangles of equal altitude are proportional to their bases"; and its converse "If the areas of two triangles are proportional to their bases, their altitudes are equal."





Let internal or external bisector meet the base or base produced in D, and let DE, DF be perp. to AB and AC respectively.

To prove $$BD : DC = AB : AC.$$

Proof. $\because$ $DE = DF$ (locus th.); $\therefore$ $\dfrac{\triangle ABD}{\triangle ACD} = \dfrac{AB}{AC}$,

but $$\frac{\triangle ABD}{\triangle ACD} = \frac{BD}{DC} \text{ (common vertex } A);$$

$$\therefore \quad AB : AC = BD : DC.$$

Conversely, if $BD : DC = AB : AC$ to prove AD bisects $\angle BAC$.

Proof. We have $$\frac{\triangle ABD}{\triangle ACD} = \frac{DB}{DC}$$

$$= \frac{AB}{AC} \text{ (by hyp.)};$$

$$\therefore \text{ altitude } DE = \text{altitude } DF,$$

i.e. AD is one of the bisectors of $\angle BAC$.

May I suggest also the use of the $\propto$ sign to denote similarity between triangles, *e.g.* $\triangle ABC$ is similar to $\triangle DEF$ would be stated thus:

$$\triangle ABC \propto \triangle DEF.$$

Boys' High School, Kimberley. R. HAMILTON DICK.

735. [A. 1. a.] *A Note on the Treatment of the Geometrical Progressions which arise in the calculation of the cost of an annuity.*

In calculating the cost of an annuity, the summation of the geometrical progression is more easily effected if we multiply each term by the reciprocal of the common ratio than if we multiply by the common ratio itself. *E.g.* to find the cost of an annuity of £10 per annum to continue for five years, money being worth 4 per cent. per annum.

$$P = \frac{10}{1{\cdot}04} + \frac{10}{(1{\cdot}04)^2} + \frac{10}{(1{\cdot}04)^3} + \ldots + \frac{10}{(1{\cdot}04)^5};$$

$$\therefore \quad P(1{\cdot}04) = 10 + \frac{10}{1{\cdot}04} + \frac{10}{(1{\cdot}04)^2} + \frac{10}{(1{\cdot}04)^3} + \frac{10}{(1{\cdot}04)^4};$$

$\therefore$ by subtraction, we have:

$$P({\cdot}04) = 10 - \frac{10}{(1{\cdot}04)^5}, \text{ etc.}$$

The more orthodox method gives:

$$P\left(1 - \frac{1}{1{\cdot}04}\right) = \frac{10}{1{\cdot}04} - \frac{10}{(1{\cdot}04)^6}.$$

This method is useful in solving ordinary Geometrical Progressions where the common ratio is less than unity. J. W. BROOKS.

736. [V. 1. a. μ.] *The Stroud System.*

I am very glad to see that the complete specification of the quantities involved in an applied mathematical investigation is being systematically insisted on in teaching students of physics and mechanics. I have always held that the formulae of mechanics refer to the complete quantities, and not merely to their numerical values in some conventional system of units, *e.g.* " g "$\simeq \dfrac{32 \text{ feet}}{\text{sec}^2} = 981 \dfrac{\text{cm}}{\text{sec}^2}$; not $g=32$ in the British system and $g=981$ in the C.G.S. system. The latter statements are true, but are detached fragments, and often lead to such lamentably erroneous statements as that 1 lb weight $=g$ poundals, which is found even in some text-books.

From fundamental equations, of course, useful numerical formulae can be evolved; for example, in the case of visible distance x miles of horizon at sea from a masthead y feet high, the fundamental approximate equation is $d^2=2Rh$,

$$\therefore\ (x \text{ miles})^2 = (7920 \text{ miles}) \times y \text{ feet},$$

$$\therefore\ x^2 = \frac{7920\, y}{5280} = \frac{3}{2} y.$$

In this numerical equation x is not a distance, it is a ratio, *viz.* the ratio of the visible range to a mile $= \dfrac{\text{range}}{\text{mile}}$, and $y = \dfrac{\text{height}}{\text{foot}}$. This is true of all strictly numerical formulæ; each letter is the ratio of the quantity under consideration to some specified unit.

There is one important matter which should be considered in connection with the Stroud system, *viz. direction*, when the formula refers to vectors.

In the absence of $\sqrt{-1}$, all terms in an equation must not only be of equal dimensions, but also, if they are vectors, they *must be co-directional.*

Thus, the horizontal range of a projectile

$$= \frac{2\, V^2 \sin \alpha \cos \alpha}{g}.$$

On analysing this,

$$V \sin \alpha = \frac{\text{Vertical length}}{\text{time}}, \quad V \cos \alpha = \frac{\text{horizontal length}}{\text{time}}$$

$$g = \frac{\text{vertical length}}{(\text{time})^2}.$$

$\therefore$ on inserting these in the formula, we find it *necessarily* represents a horizontal length, even if we had not known the fact to start with.

A. LODGE.

Sir J. B. Henderson's article on the 'Stroud' system of dynamics is valuable and timely, and his suggestion that it ought to be used in schools is a natural corollary of the conclusion he has reached from experience with more advanced students.

Those of us who have been accustomed to think that the symbols of algebra are applicable to numbers only can reconcile ourselves to the 'Stroud' system by taking the *ft*, *lb*, and *sec* and *Pound-weight* of the 'Stroud' formula: "1 Pound weight $= 1 \text{ lb.} \times 32 \dfrac{\text{ft.}}{\text{sec.}^2}$" to represent pure numbers, *viz.* the *measures* of the foot, etc., in terms of *any* absolute set of units, *i.e.* such that the unit of force acting on the unit of mass gives it unit acceleration. This gets over the difficulty of defining the meaning of the signs $\times$ and $\div$ as applied to *rates.*

This remark applies also to the notation used by Clerk Maxwell in his great treatise, where he deals with the theory of dimensions. Clerk Maxwell defines $[L]$ as the concrete unit of length, so that an actual length would

be fully expressed by $l\,[L]$, where l is the measure of the length, *i.e.* its ratio to the unit. If we take $[L]$ to be the *measure* of the actual unit in terms of a fundamental unit (which does not need to be specified), then the dimensional equations of that author can be interpreted as equations involving pure numbers only, and his statement that "the dimensions of the unit of velocity are $[L\,T^{-1}]$," which he does not explain, is interpreted to mean that the measure of the unit of velocity in terms of a 'fundamental unit' is the quotient of the corresponding measures for length and time.* The idea put forward in this note will not appeal to all, to some it may seem an unnecessary assistance to "common sense," but I hazard it in the belief that others besides myself have wished for a clearer logical basis for the "theory of Dimensions."

R. F. MUIRHEAD.

737. [K[1]. 9. a.] *Note on the Construction of a Regular N-gon through N given points.*

A full enunciation of the problem is, "Given n points which taken in order form the vertices of a convex polygon of n sides, to describe, if it be possible, a regular polygon of n sides, on each of which there lies one of the points, points and sides being taken in the same order: to determine the number of and a geometric expression for the necessary and sufficient conditions."

The solution depends on the principle that if an angle XPY, where X and Y are fixed points on a circle and P a variable point on one of the arcs XY, be divided internally in the ratio $p:q$, then the dividing line divides the supplementary arc in the ratio $p:q$. It is supposed that the divisions of arcs in the subsequent construction are obtained from this property, Euclidean constructions for the division of angles not being demanded.

Starting with a pair of adjacent points denote the given points in one direction by 1, 2, 3 ..., and in the other by 1′, 2′, 3′ ..., as indicated in the figure (for the case $n=9$). On pairs of points (1, 1′), (2, 3), (4, 5), (2′, 3′), etc., describe circles whose arcs external to the polygon contain angles $\frac{n-2}{n}\,180^\circ$ (the angle of the required figure). The pairing of the points is shown on the table by brackets: the variations in the final circles in the systems for different forms of n appear logically in actual construction. Divide the supplementary arc on (1, 1′) into $n-2$ equal parts by the points $\gamma, \delta, \epsilon \ldots, \gamma', \delta', \epsilon' \ldots$; on the supplementary arcs (2, 3), (2′, 3′) cut off arcs $2c$, $2c'$ each $\frac{1}{n-2}$ of the arcs; on the supplementary arcs (4, 5), (4′, 5′) cut off $4e$, $4e'$ each $\frac{3}{n-2}$ of the arcs; and so on. Join γc, ϵe, $\gamma' c'$, $\epsilon' e'$, etc., then it is necessary but not sufficient for the possible construction of the polygon that these lines meet on the arc (1, 1′) at A, a number of conditions 2 less than the number of circles. To obtain C produce γc to meet (2, 3) in C; having A and C, B is obtained, and similarly for the other vertices. The figure now consists of a polygon broken up into triangles and quadrilaterals with one common vertex whose angles equal those of corresponding parts of a regular polygon, any collinearities of the type E, E', $2m+1$ in a key triangle as shown in the figure being fulfilled automatically. To obtain a regular figure it is further necessary that points of type A, δ, D be collinear. In fact, attached to all

* From the point of view of the present note, the fact that the dimensions of the electrostatic unit are $\frac{L}{T}$ times those of the electromagnetic unit where $\frac{L}{T}$ is the measure of the velocity of light, is seen to mean that the *measure* of the former unit in terms of an unspecified fundamental unit is $\frac{L}{T}$ times the measure of the latter, *i.e.* on the C.G.S. system, a large number, so that on that system the former unit is the smaller, since the magnitude of two quantities is inversely as their measures.

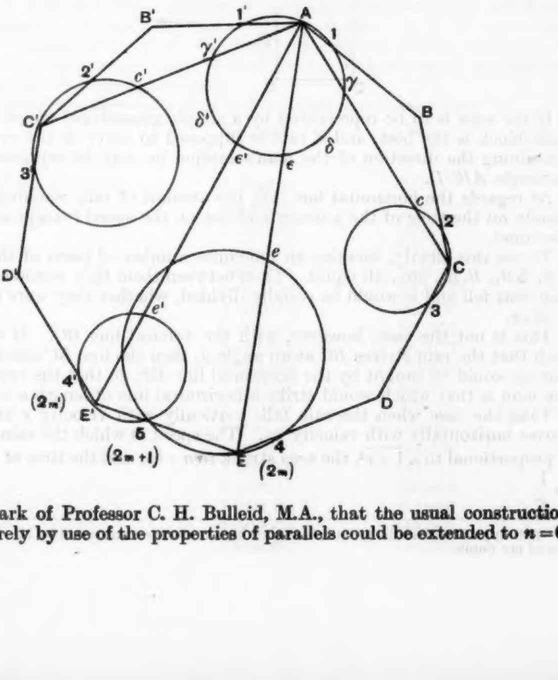
but one of the $n-3$ points γ to γ' is a condition, or $n-4$ in all, and these conditions are not only necessary but are also sufficient.

The table gives, in addition to the scheme of circles, their number, and the numbers of triangles and quadrilaterals formed. The number of conditions demanded for the possible construction of a regular n-gon is the sum of the number of circles less 2 and the number of quadrilaterals, which will be seen to be $n-4$ in each case.

$$
\begin{array}{cccc}
4m & 4m+1 & 4m+2 & 4m+3 \\
\overbrace{1\quad 1'} & \overbrace{1\quad 1'} & \overbrace{1\quad 1'} & \overbrace{1\quad 1'} \\
\left\{\begin{matrix} 2 & 2' \\ 3 & 3' \end{matrix}\right\} & \left\{\begin{matrix} 2 & 2' \\ 3 & 3' \end{matrix}\right\} & \left\{\begin{matrix} 2 & 2' \\ 3 & 3' \end{matrix}\right\} & \left\{\begin{matrix} 2 & 2' \\ 3 & 3' \end{matrix}\right\} \\
\cdots & \cdots & \cdots & \cdots \\
\left\{\begin{matrix} 2m-2 & 2m-2' \\ 2m-1 & 2m-1' \end{matrix}\right\} & \left\{\begin{matrix} 2m & 2m' \\ 2m+1 & 2m+1 \end{matrix}\right\} & \left\{\begin{matrix} 2m & 2m' \\ 2m+1 & 2m+1' \end{matrix}\right\} & \left\{\begin{matrix} 2m & 2m' \\ 2m+1 & 2m+1' \end{matrix}\right\} \\
\underbrace{2m\quad 2m'} & & & \left\{\begin{matrix} 2m+2 & 2m+2 \\ 2m+1' & 2m+1 \end{matrix}\right\}
\end{array}
$$

No. of	$4m$	$4m+1$	$4m+2$	$4m+3$
Circles	$2m$	$2m+1$	$2m+1$	$2m+3$
Triangles	2	3	2	5
Quadrilaterals	$2m-2$	$2m-2$	$2m-1$	$2m-2$
Conditions	$4m-4$	$4m-3$	$4m-2$	$4m-1$

If in any actual construction it be assumed that the polygon is possible, or if checking be allowable by direct measurements of the resulting figure, the drawing work may be very greatly simplified, as two consecutive circles (1, 1′) (2, 3) are sufficient to give a side, but it seems desirable to obtain some method which gives the required conditions in a more descriptive manner.



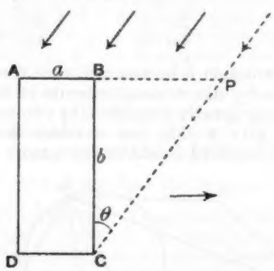
To a remark of Professor C. H. Bulleid, M.A., that the usual construction for $n=4$ purely by use of the properties of parallels could be extended to $n=6$,

I am indebted for a recognition that this method is equally applicable to any $n=2m$. In the given irregular polygon the join of points x, $m-x+1'$ in our notation is a diameter. Through the base point 1 a line equal in length to this is drawn in a direction "into the polygon" at an angle $\frac{180(x-1)}{m}$ to the diameter. Then the other extremity lies on the side of the required regular polygon (if it can exist) opposite to that through 1; its direction is thus obtained, and the polygon may then be completed. The necessary condition that all the points so obtained for x, 2 to m, should be collinear with m', ensures that opposite parallel sides are at the same distance apart; it does not, however, ensure that all sides are equal, and from the point of view that all the conditions have not been tested the solution is incomplete. H. G. GREEN.

University College, Nottingham.

738. [R. 1. d.] *A Rain Problem.*

The capacity of rain for doing damage, for instance, in wetting clothes through, is certainly increased if its velocity is increased. Let us suppose that its effect is proportional to the square of the velocity* and let us consider at what speed a man should walk (or run) so as to have the best chance of not being wet through in going a given distance.



If the man is to be represented by a simple geometrical figure, probably a right-block is the best, and if rain is supposed to move in the vertical plane containing the direction of the man's motion he may be represented by the rectangle $ABCD$.

As regards the horizontal line AB, the amount of rain reaching it depends merely on the time of the journey and not on the speed (except as this alters the time).

To see this clearly, imagine an indefinite number of parts of the same line AB, BB_1, B_1B_2, etc., all equal. Then between them they would catch all the rain that fell and it would be equally divided, whether they were moving fast or slow.

This is not the case, however, with the vertical line BC. If the speed is such that the rain strikes BC at an angle θ, then the line BC catches as much rain as would be caught by the horizontal line BP, so that the rain caught by the man is that which would strike a horizontal line of length $a+b\tan\theta$.

Take the case when the rain falls vertically with velocity v and the man moves horizontally with velocity xu. The speed at which the rain strikes him is proportional to $\sqrt{1+x^2}$, the area struck to $a+bx$, and the time of the journey to $\frac{1}{x}$.

* This hypothesis and the problem to which it gives rise were suggested by Mr. R. S. L. Baker, one of my pupils.

The expression which has to be made minimum is therefore

$$(a+bx)(1+x^2)/x \quad \text{or} \quad ax^{-1}+b+ax+bx^2.$$

This is minimum if $\quad a+2bx-ax^{-2}=0,$

or if $\quad 2bx^3+a(x^2-1)=0.$

The ratio of a to b will differ with the direction in which the man has grown best, but perhaps $b/a=5$ might serve as a not unusual case, so that

$$10x^3+x^2-1=0.$$

This equation gives

$$x=\cdot 43,$$

as is shown at the side.

10	1	0	-1
1	1	0	-100
	4	20	80
	5	20	-20,000
	4	36	
	9	56,00	
	4		
	13,0		

The man must therefore walk at somewhat less than half the rate at which the rain is falling to get least wet.

If b/a is taken as 4 we get $x=\cdot 46$, so we have the consoling thought that the fatter the man the faster he should walk.

Again, if the problem were somewhat varied: if it were a question of carrying an uncovered basket of perishable fruit, so that the correct representation would be the horizontal line only, putting $b=0$ (or working independently) we get $x=1$. So that the man carrying fruit must run or somehow get along twice as fast as the man who is only trying to save his own skin. The complication of the problem can be increased by supposing the rain to fall at an angle with the vertical, the wind blowing from one side, but the precarious nature of the hypothesis as to the effect of the rain will probably prevent the results from having much scientific value. C. O. TUCKEY.

739. [A. 1.] I do not know if there is anything new about the following method of making arithmetically one's own log. tables. It is, of course, an adaptation of methods by Prof. Perry and others, but I have not seen it worked out in quite the same way. I have tried it with several of my classes, and it has given very accurate results.

Taking successive square roots of 10 and tabulating results, we get

		Number.	Log.
$\sqrt{10}=3\cdot 162278,$	$\therefore$	3·162278	·5
$\sqrt{3\cdot 162278}=1\cdot 778280,$	$\therefore$	1·778280	·25
etc., etc.		1·333523	·125
		1·154784	·0625
		1·074610	·03125
		1·036636	·015625
		1·018154	·007812
		1·009036	·003906
		1·004509	·001953
		1·002252	·000977
		1·001125	·000488
		1·000562	·000244
		1·000281	·000122
		1·000140	·000061
		1·000070	·000030
		1·000035	·000015
		1·000017	·000008
		1·000008	·000004
		1·000004	·000002
		1·000002	·000001

To find the log. of any number (say 2), divide in succession by the highest power of 10, *e.g.*

$$\frac{2}{1\cdot778280}=1\cdot124682,$$
$$\frac{1\cdot124682}{1\cdot074610}=1\cdot046597,$$
$$\frac{1\cdot046597}{1\cdot036636}=1\cdot009609,$$
$$\frac{1\cdot009609}{1\cdot009036}=1\cdot000568,$$
$$\frac{1\cdot000568}{1\cdot000562}=1\cdot000006,$$
$$\frac{1\cdot000006}{1\cdot000004}=1\cdot000002,$$
$$\frac{1\cdot000002}{1\cdot000002}=1.$$

Add up logs. of divisors

$$\begin{array}{r}
\cdot25 \\
\cdot03125 \\
\cdot015625 \\
\cdot003906 \\
\cdot000244 \\
\cdot000002 \\
\cdot000001 \\ \hline
\cdot301028
\end{array}$$

$= \cdot30103$ to 5 places.

Simon Langton School, Canterbury. P. L. HALL.

740. [X. 8.] *Duplication, Trisection, and the Elliptic Compasses.*

(I.) Circle-squaring apart—the duplication of the cube and the trisection of the angle have come down to us as the two most famous geometrical problems of antiquity. They still arouse both our curiosity and our interest.

But on approaching these problems closer, the student may well feel discouraged. Their solution cannot be effected by the ruler and compasses of Euclid. Not only must he arm himself with new and varied apparatus, but he must embark on fresh studies of the conic sections or, perhaps, of such "fancy" curves as the cissoid and the conchoid.

Valuable such study can undoubtedly be; but it demands more time than is available in all cases and on all occasions. Consequently there is ample excuse for mention of an instrument which, involving no curve other than the circle, yet solves both our problems with simplicity and speed.

We allude to the Elliptic Compasses.

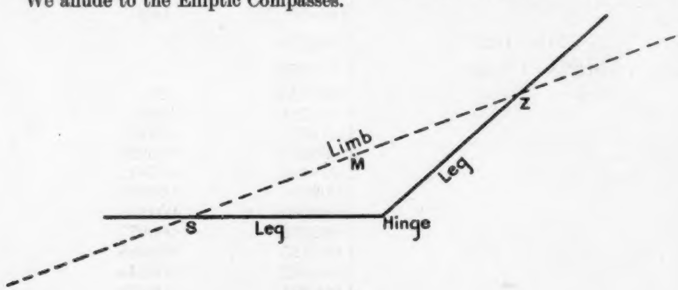


FIG. 1.

(II.) The Elliptic * Compasses. (Fig. 1.)

This instrument is made up essentially of:

(1) The Frame—consisting of two "legs," slotted longitudinally, and hinged together so as to be adjustable at any given angle to one another.

* So called because as *S* and *Z* move along the legs, every other point in the limb describes an ellipse.

(2) The Limb—a movable ruler, fitted with two studs—S and Z—gearing into the slots of the frame; the length SZ being adjustable at will, and M (the midpoint of SZ) being clearly marked and recognisable.

We postulate our ability to recognise when any point on the Limb—moving under control of the Frame—coincides with any given point within its range. (To some this "ability" may appear implicit in Euclid's First and Second Postulates—where it is recognised in the case of *two* points. But it is well to put the matter beyond a doubt. We thus preclude the necessity for consideration of other curves.)

(III.) The duplication of the cube identifies itself with the finding of two mean proportionals between two given lengths, if one length—let us take the smaller—is treated as unity. It is, indeed, well to commence by noting that if from a point outside a circle (Fig. 2) two lines c and d be drawn cutting the circle in chords a and b; and if $ab=cd$, then from *Euclid* III. 36, *Cor.*, it follows that $ac=d^2$, and $bd=c^2$, and

$$a : d :: d : c :: c : b.$$

And, again, if we put

$$b=1,$$

then c and d are expressible in terms of a, as

$$c=a^{\frac{1}{3}},$$

$$d=a^{\frac{2}{3}}.$$

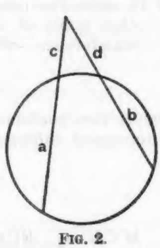


FIG. 2.

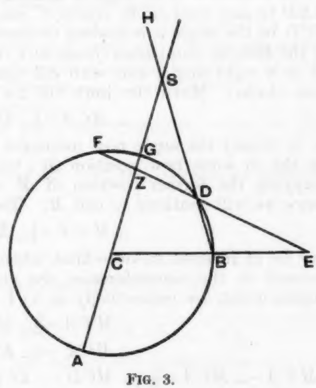


FIG. 3.

Consequently, if we wish to find the cube root of some line, greater than that we have selected as unity, we must proceed as follows (Fig. 3):

With centre C describe a circle ABD on a diameter equal to the given line. In it place the chord BD equal to unity. Join CB and produce to E so that CE equals the diameter. Join ED and produce to F. Produce BD to H. And now apply the Elliptic Compasses [hinge at D, and legs along DH and DF], adjusted to an angle equal to FDH, and with SZ equal to BC or DE. Move the limb till its prolongation passes through C, cutting the circle in G and A.

Now (by Menelaus)

$$SZ \cdot CE \cdot BD = CZ \cdot BE \cdot SD \text{ (and } SZ=BE\text{)};$$

therefore $$CE \cdot BD = CZ \cdot SD, \text{ or } AG \cdot BD = SG \cdot SD;$$

and as BD represents unity

$$SG = AG^{\frac{1}{3}},$$

$$SD = AG^{\frac{2}{3}}, \text{ as above.}$$

Note.—This construction is adapted from that stimulating little book, Sundara Row's *Paper Folding* (Addison & Co., Madras, 1893; and Open Court Co., 1901.)

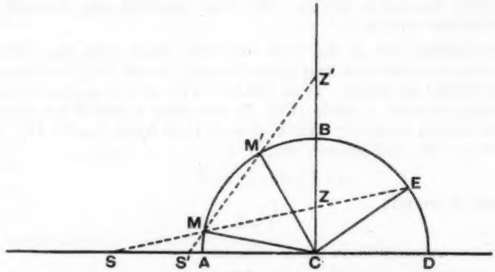


FIG. 4.

(IV.) The trisection of an angle is effected as follows (Fig. 4).

Let ABD be any semi-circle, centre C, and BC an ordinate.

Let ECD be the angle it is desired to trisect.

Apply the Elliptic Compasses (hinge at C, and legs along CA and CB—thus) adjusted to a right angle, and with SZ equal to AD. (M will in this case follow the circle.) Move the limb till its prolongation passes through E. Then

$$\angle MCA = \tfrac{1}{3}\angle ECD.$$

Again, to trisect the angle now indicated by MCD, move the limb till its midpoint lies in some new position M', but some other point of the limb now occupying the former position of M on the circumference—which for convenience we will continue to call M. Then

$$\angle M'CA = \tfrac{1}{3}\angle MCD.$$

(It may be of interest to note that when, as above, two positions of the limb intersect on the circumference, the chords intercepted subtend at the centre angles which are respectively as 3 : 1.

For

$$\angle M'CA = \tfrac{1}{3}\angle MCD$$

$$\angle MCA = \tfrac{1}{3}\angle ECD.$$

$$\angle M'CA - \angle MCA = \tfrac{1}{3}\ [\angle MCD - \angle ECD] \text{ or } \angle M'CM = \tfrac{1}{3}\angle MCE.)$$

Taking the radius of the circle as unity, this figure affords very direct and easy proof of the trigonometrical formulae for $2\cos 2\theta$, and $2\cos 3\theta$ in terms of $2\cos\theta$, etc.

(V.) We have shown how the Elliptic Compasses can deal with both our problems; and a word on the Algebra underlying may not be out of place.

Every rational cubic equation is reducible to the form

$$x^3 + \frac{1}{x^3} = \pm A$$

where we wish to find x, and where A is some numerical value involving only rational numbers and square roots.

If $+A > +2$, it is seen readily that the equation leads to a duplication solution as above.

If $+A < +2$, it can be regarded as $2\cos\phi$;

whence
$$x+\frac{1}{x}=2\cos\tfrac{1}{3}\phi,$$
and we can proceed by trisection.

(VI.) The solution of the quadratic equation is part of our daily life and our civilization, and it is impossible to overrate the services of Euclid as its geometrical interpreter.

But the cubic is on an altogether different footing. Still something of a toy, it is a rebus for the study rather than a tool for everyday work. We really have no knowledge whatever of its full potentialities. Indeed, it seems probable that we have not yet fully mastered its "idioms." What it lacks is the polish—the handling—of *common use*!

To compare the quadratic and the cubic may be idle, but at least one contrast is significant and may be permitted. Between 5 and 257 only *one* prime number, 17, lends itself to cyclotomy by quadratics. Admitting the cubic, *nine* more primes—7, 13, 19, 37, 73, 97, 109, 163, 193—all become tractable to this end.

Might it not be well to admit the Elliptic Compasses to general use, and to include the cubic equation more generally in our curriculum?

The brevity and—I trust—the simplicity of this paper indicate that such "new learning" need absorb but very little time.

85 Wilbury Crescent, Hove, Sussex, England. C. H. CHEPMELL.

741. [R. 5. a.] *Notes on the Potential of an Ellipsoid.*

In the methods given in most text-books the determination of (1) the length elements in the ellipsoidal coordinates and (2) the capacity of an ellipsoid charged to a given potential are somewhat lengthy. The following methods of effecting these determinations may be of interest.

(1) Determination of Length Element.

The Cartesian coordinates of a point are connected with its ellipsoidal coordinates by means of
$$x^2=(a^2+\lambda)(a^2+\mu)(a^2+\nu)/(a^2-b^2)(a^2-c^2), \text{ and two similar equations,}$$
$$=A_\lambda^{-1}A_\mu^{-1}A_\nu^{-1}/D_{a,b}D_{a,c} \text{ (say).}$$
From these we derive
$$2\partial x/\partial\lambda = xA_\lambda;\quad 2\partial y/\partial\lambda = yB_\lambda;\quad 2\partial z/\partial\lambda = zC_\lambda.$$
Hence if $h_1 d\lambda$ is the length element corresponding to $d\lambda$,
$$4h_1^2=\frac{x}{a^2+\lambda}\cdot\frac{x}{a^2+\lambda}+\frac{y}{b^2+\lambda}\cdot\frac{y}{b^2+\lambda}+\frac{z}{c^2+\lambda}\cdot\frac{z}{c^2+\lambda}=\sum_{ABC}^{xyz} xA_\lambda\,.\,xA_\lambda. \tag{1}$$
Since the surfaces λ = constant are orthogonal to the surfaces μ = constant and ν = constant,
$$0=\Sigma xA_\mu\,.\,xA_\lambda, \text{ ....} (2)$$
$$0=\Sigma xA_\nu\,.\,xA_\lambda. \text{ ....} (3)$$
On solving (1) (2), (3) as equations for $x/(a^2+\lambda)$, we obtain
$$4h_1^2yz=\begin{vmatrix} B_\mu & C_\mu \\ B_\nu & C_\nu \end{vmatrix}=yz\,.\,A_\lambda\begin{vmatrix} x^2A_\lambda & B_\lambda & C_\lambda \\ x^2A_\mu & B_\mu & C_\mu \\ x^2A_\nu & B_\nu & C_\nu \end{vmatrix}=yz=A_\lambda\begin{vmatrix} 1 & B_\lambda & C_\lambda \\ 1 & B_\mu & C_\mu \\ 1 & B_\nu & C_\nu \end{vmatrix}$$
(since $\Sigma x^2A_\theta=1$ when $\theta=\lambda,\mu,\nu$)
$$=yzD_{\lambda,\mu}D_{\lambda,\nu}A_\lambda B_\lambda C_\lambda\begin{vmatrix} 1 & 1 & 1 \\ 0 & B_\mu & C_\mu \\ 0 & B_\nu & C_\nu \end{vmatrix}$$

(by subtracting the first row of the determinant from the second and third rows).

Therefore $$h_1^2 = \tfrac{1}{4} D_{\lambda,\mu} D_{\lambda,\nu} A_\lambda B_\lambda C_\lambda.$$

(2) Capacity of Ellipsoid.

By the usual method it is shown that the potential in the field of the ellipsoid $\lambda = 0$ raised to a potential V_0 is given by

$$V = C \int_\lambda^\infty dt (A_t B_t C_t)^{\frac{1}{2}}, \qquad \text{.............................(1)}$$

where $$C = V_0 \Big/ \int_0^\infty dt (A_t B_t C_t)^{\frac{1}{2}}.$$

The labour involved in finding the charge E from the equations

$$E = \iint \sigma dS \quad \text{and} \quad -4\pi\sigma = (h_1^{-1} \partial V / \partial\lambda)_{\lambda=0}$$

can be avoided by the following artifice:

At a great distance R from the ellipsoid the potential is approximately E/R.

Furthermore λ is approximately R^2, t in the integral in (1) is large compared with a^2, b^2, c^2, and (1) gives the approximate equation

$$V = C \int_{R^2}^\infty t^{-\frac{3}{2}} dt = 2C/R.$$

Comparison of the two approximations gives $E = 2C$. Hence the capacity

$$E/V_0 = 2C/V_0 = 2 \Big/ \int_0^\infty dt (A_t B_t C_t)^{\frac{1}{2}}.$$

University of Queensland. H. J. PRIESTLEY.

742. [X. 10. b.] *Despiau's Select Amusements.*

One of my colleagues possesses L. Despiau's *Select Amusements in Philosophy and Mathematics*, London, G. Kearsley, 1801, pp. 397, which is not listed by Ahrens * and Lucas † in their bibliographies on mathematical recreations.

The book is recommended by Dr. Hutton, and is a translation from a French edition, *Choix d'amusements physiques et mathématiques*, Londres, Dulau, 1800, 2 vols. in-12°, listed in Quérard's *La France littéraire*, and in Watt's *Bibliotheca Britannica*. A copy of the English edition is in the British Museum, but otherwise these books appear to have escaped the nets of the greater libraries of Great Britain, France and America. Both editions are reviewed in the *Monthly Review*, Vol. 37, London, 1802, pp. 92, 93. We have found nothing about Despiau, except that he styled himself a professor of mathematics and physics "at London" (in the French edition), and "formerly at Paris" (in the English edition), and that he was the author also of *Usage des globes*, London. Possibly some readers possess additional information. Despiau begins his book with an argument on the necessity and use of mental recreation. Amusements "are remedies invented to revive depressed spirits, and to render the mind capable of resuming its usual labours with greater success; but a wise man will employ them with moderation." "In compiling the present collection, Ozanam, and those who have written on the same subject, have been my guides; . . . these amusements are the production neither of one man, nor one age, but of a great number of the learned, of artists, and of many ages of research and of observation."

University of California. FLORIAN CAJORI.

* W. Ahrens' *Mathematische Unterhaltungen und Spiele*, Leipzig, 1901, pp. 403-419.

† Édouard Lucas' *Récréations mathématiques*, vol. 1, 2 éd., Paris, 1891, pp. 237-248.

REVIEWS.

Report on Radiation and the Quantum Theory. By J. H. Jeans. Second edition. 7s. 6d.; bound in cloth, 10s. 6d. (Physical Society Reports. London: The Fleetway Press.)

On the Application of the Quantum Theory to Atomic Structure. Part I.: The Fundamental Postulates. By Niels Bohr. From the *Zeitschrift für Physik*, 13, 117 (1923). Translated by L. F. Curtiss, National Research Fellow (U.S.A.). 3s. 6d. (Proceedings of the Cambridge Philosophical Society, Supplement. Cambridge University Press.)

The Physical Society has been happy in the choice of subjects for its *Reports*. Each has become justly famous as soon as published; each has contained exactly the information that students and workers in the subject needed, and in each the presentation has been convenient and attractive. Jeans' *Report on Radiation and the Quantum Theory*, published in 1914, was for long the only account of the subject available in English. It was moreover a sympathetic account, and it was written by one who had for years been in the front line in the battle in which the classical theory, as claiming to be capable of providing a satisfactory explanation of the facts of radiation, had been overthrown by the quantum theory. In the intervening ten years, the quantum theory has gone swiftly from success to success, and the appearance of a second edition of Jeans' *Report* fittingly marks the completion of the decade.

Readers of a review of a second edition may reasonably expect to be told exactly how the new edition differs from the old. The new *Report* contains roughly the same number of pages as the old, but the pages are the larger ones used in A. Fowler's *Report on Series in Line Spectra*. Thus there is a considerable net addition of material. As remarked in the preface, the first edition had to be an *apologia* as well as an exposition, but this is no longer necessary. Consequently such passages as the quoted remarks of various authorities at the Solway Conference of 1911 have now been omitted. The principal addition is an entirely new chapter of 17 pages, entitled, "The Dynamics of the Quantum Theory," but other additions have been made throughout, and the last chapter on the physical basis of the quantum theory has been entirely re-written. The space given to atomic structure and the modern theory of spectra is not proportional to its importance, but this is not the main province of the book.

There is a curious tendency in some treatises on physics to refer to certain important investigations made in the past as being now "chiefly (or only) of historic interest." The statement would appear to imply that such investigations are the concern of the antiquary rather than of the physicist. We may instance Wien's displacement law, which gives the functional form of the intensity-distribution in black radiation, and which is said to be superseded by Planck's law. It is difficult to learn what these writers mean by the phrase we have put in inverted commas. There exist, of course, in large numbers, imperfect investigations which have been completed as physics has evolved, and to pioneer investigations of this kind the phrase may rightly be applied. But an investigation such as that which leads to Wien's law is a permanent achievement, representing the utmost to which deductions based on a given field of data and hypothesis can be carried. It constitutes one of the fine fruits of what is called mathematical physics. It is one of the things we want to know, and are glad to know. To say that such are "only of historic interest" is to make a distinction between these and other results which is not the distinction on which stress should be placed. Some results constitute a terminus of endeavour; others are the bridges or other constructional triumphs of the route. To dismiss the former as "only of historic interest" is to take at bottom a utilitarian and un-aesthetic view of the progress of science.

Of recent years there has been a tendency to treat in this way those investigations which showed that Maxwell's equations implied a partition of radiant energy which is in disagreement with the observed partition. The second edition of Jeans' *Report* rightly gives them the same prominence which they had

in the first. Planck's investigation of the equilibrium between radiation and resonators, Jeans' investigation of the equilibrium between radiation and free electrons, Lorentz and Jeans' investigations of the equilibrium between radiation and electrons in conductors—all are still as important to-day, as exposing the consequences of the classical theory, as they were when first published. If one of them were lost, it would be necessary for it to be done over again. They happen to be rather difficult, and it seems appropriate here to pay a tribute to Jeans' work in this region.

In this connection it is interesting to notice a curious lacuna in the development of the quantum theory. When these investigations had shown that something was wrong, a revised dynamics became necessary. Unsatisfactory though his assumptions were, Planck did in fact reproduce his investigation in a quantum form that led to the observed partition of energy. Planck's *result* was then accepted, but for many years no one bothered to reproduce Jeans' and Lorentz' investigations in quantum form. This was partly owing to Jeans' deduction of the radiation law from consideration of the aether in an enclosure alone, without the intervention of matter. But the old problem, of how free electrons or electrons in conductors could set up the observed partition of energy, still remained unsolved. Within the last year, Pauli has formulated rules for the interaction of free electrons and radiation which lead to Planck's law for an electron-gas in equilibrium, though the problem of the interaction of free electrons in a conductor and radiation still invites solution. Since we know that the interaction must lead to Planck's law, the solution will be a contribution to the theory of conductors rather than to the theory of radiation. A fourth problem, that of the interaction of radiation and Bohr atoms, was dealt with by Einstein in 1917 in a masterly investigation which is fully treated by Jeans in the new edition and which is the inspiration of many papers at the present time.

In the additional chapter, Jeans gives a brief but attractive account of the work of Bohr, Sommerfeld, Ehrenfest, Burgers and others in developing the laws of quantum dynamics in a form capable of application to dynamical systems of varying degrees of generality. An account is given of the correspondence principle, the adiabatic principle, and the quantising of conditionally periodic systems for which "separation of the variables" can be carried out, with examples. This chapter leads us naturally to the consideration of the second of the works under review.

Bohr's essay is neither an original paper nor a formal treatise, in the ordinary sense of these terms. It has the character of a charter, of a legal document whose aim is to put in statutory form the rules (according to Bohr) by which the quantum theory can be used to give a description of natural phenomena. Indeed it has in some ways the verbal form of a legal document, with its prolixity and repetition and its use of heavy self-contained sentences. A greater contrast than between the styles of Jeans and Bohr could hardly be imagined. Jeans has the art of carrying his reader enthusiastically along with him. Bohr recites a series of sentences, the words of which are so chosen that each sentence will be independently true for all time. Owing to the uncertainties of the quantum theory, this is often not possible without the use of vague and sometimes obscure language, and moreover many of the sentences are packed with reservations and recapitulations. The translator's task has been one of great difficulty, but the wrapped-up nature of the sentences is not due to any failure to anglicise German modes of expression. The style is reminiscent of that of Willard Gibbs. Gibbs, in his great memoir on the equilibrium of heterogeneous substances, did indeed succeed in writing sentences true for all time, for they remain just as true to-day now that there are so many then-unknown phenomena which are covered by them. Bohr's task is harder, in virtue of the formlessness of the quantum theory as compared with thermodynamics.

Such being the characteristics of Bohr's style, it does not lend itself to the development of detailed arguments, and the essay consists of statements rather than of arguments. Much work, often mainly tentative, has been done on the Continent in the last few years on the quantum developments of analytical dynamics This essay gives a formal re-statement of those of the results

which Bohr believes to be a self-consistent accurate description of what actually occurs in nature. Those results of which he disapproves are mentioned in copious footnotes. His objections frequently amount to less than complete refutations; they are often concerned less with the success or non-success of a particular attempt than with its position in the comprehensive scheme of quantum dynamics to which Bohr is striving. Even where something of the nature of a deduction or an argument is being put forth, it is not always clear which statements follow immediately from their predecessors, which can be shown so to follow, and which are fresh.

It will thus be seen that the essay will be of little use to those not already fairly familiar with dynamics and the quantum theory. But there is comparatively little mathematics, and practically nothing in the way of mathematical analysis, though mathematics is freely used descriptively.

Nevertheless the essay accomplishes in the end a great gain of clarity. One feels that Bohr is thinking more profoundly about these problems than other people. He is not content with little scraps of explanation and success, here and there. He insists on purifying and generalising the principles which surround the application of the fundamental postulates, and in giving them greater elasticity, so that even their present vague form shall include nothing incompatible with their ultimate precise content.

For Bohr has an almost wistful belief that a complete set of rational principles does exist, if only we could find it. Perhaps everyone believes this, but in Bohr's hands the belief becomes almost part of the technique of investigation. For example, it is usually stated that the orbits of the electrons in an atom are those calculated on the classical theory, if the classical radiation is neglected; it is supposed, as it were, that in the stationary states, quantum theory is just classical theory with the radiation left out. But for Bohr, it is only that the motion in the stationary states can be described in this way *to a close approximation*. When the classical radiation would be large, the approximation ceases to be close. In some cases, as in the sudden interaction of two atoms or of an atom and an electron, the classical theory gives no approximation. In other cases, as in the emission of radiation in wireless telegraphy, which is governed by the classical theory, it is not that the quantum theory has broken down, or even that it is agreeing asymptotically for large quantum numbers, but that the formulation of the postulates of the quantum theory, which is necessarily effected with the aid of conceptions borrowed from the classical theory, has become invalid. Indeed Bohr bids us be prepared for a breakdown of results calculated on the classical theory, even concerning stationary states, in considering for example a perturbed motion in which a new period manifests itself which is large compared with the periods of the undisturbed motion, or in considering an atomic system which is not a multiply periodic system.

The essay consists of three chapters, the first on the stationary states, the second on the process of radiation, and the third on the formal nature of the quantum theory. The first begins with the first postulate, the general statement of the quantising of a multiply periodic system. This is more general than Jeans' account, for the latter deals only with systems to which the method of separation of variables is applicable. The number of quantum conditions is equal to the degree of periodicity, which in the case of a degenerate system is less than the number of degrees of freedom. Systems in the presence of an external conservative field of force are next discussed, and it is shown how the external field may give rise to new periods. Systems in the presence of an external field varying with the time next occupy attention. Fields varying rapidly with the time may be expected to yield motions which cannot be described by means of classical electrodynamics; fields varying slowly with the time lead to the development of the Adiabatic Principle. The chapter concludes with an account of the "weights" of the stationary states, both from a statistical and from a dynamical point of view.

The second chapter begins with the second postulate, which is usually known as Bohr's frequency relation. It then gives a statement of the Correspondence Principle, and discusses the bearing of this principle on the fixation of stationary states and the nature of radiation. The interesting question of

the frame of reference in which the frequency given by Bohr's relation is to be measured is discussed, but no definite conclusion is reached. The sharpness of fixation of stationary states is considered, and the validity of the second postulate as regards the radiation from non-isolated systems is questioned.

The last chapter has sections on the hypothesis of light quanta, the "coupling principle," the phenomena of reflexion and dispersion, and the conservation of energy and momentum. The latter principles are held to be true only statistically, and Bohr is evidently disinclined to believe that an emitting atom is endowed with recoil momentum, as Einstein has deduced on certain assumptions. By the "coupling principle" Bohr means a point of view by which the aether in an enclosure may be supposed to constitute a dynamical system, existing in stationary states and capable of exchanging energy with other dynamical systems. In this way the second postulate becomes a particular case of the first postulate, since the aether and any matter in the enclosure will obey the rules for the interaction of two non-radiating atomic systems—they must both be in stationary states before and after the interaction. The applicability of the coupling principle is, however, only obtained by neglecting the spreading propagation of radiation in free space, and Bohr therefore regards as questionable the advantage of the universality obtained by the coupling principle, as compared with the dualistic conception provided by the first and second postulates.

This brings us back for a moment to Jeans. Are we to suppose that the electromagnetic field in an enclosure actually occurs in the form of quantised stationary states, as supposed in Jeans' deduction of Planck's law? Only further knowledge of the nature of radiation can answer this. It may be mentioned that Bohr associates this method of deduction of Planck's law with the names of Ehrenfest and Debye, and omits that of Jeans.

Bohr insists repeatedly that the classical theory and the quantum theory are two entirely distinct sets of principles, and that no analysis between them or asymptotic agreements in the domain of large quantum numbers must be allowed to obscure this fact. And perhaps a stronger realisation of the gulf between the two theories is the chief impression one comes away with.

E. A. M.

A Short Course in Interpolation. By E. T. WHITTAKER and G. ROBINSON. Pp. 71. 5s. 1923. (Blackie and Son.)

Interpolation is a subject whose position is rather curious. It enters into all investigations in which our data are a limited number of quantitative facts selected from a practically unlimited number. And the methods used are essentially the same, of whatever kind the facts may be—whether, for instance, we are studying astronomy or physics or statistics. But the subject has not yet found a text-book home. Occasionally, under the rather superficial name of Theory of Differences, it receives some notice in a book on algebra. Sometimes there are remarks on it in a book on differential calculus. But the treatment is usually too brief and not sufficiently practical. The result is that the writers of a book on almost any subject involving interpolation find it necessary to devote one or two chapters to it. The treatment is then liable to err in the opposite direction: it is directed to the requirements of the particular subject, and also is not sufficiently rigid. There has long been need of a book that is simple, rigid, and practical—and interesting.

Whether Professor Whittaker and Mr. Robinson have met the need with this book, time will show. It contains the first four chapters of a work, already reviewed in this volume of the *Gazette* (p. 124), on the "Calculus of Observations," but this latter subject is wide enough to allow interpolation to be treated in a general way.

A curious omission is that of quadrature, which would seem naturally to go with interpolation. The omission is the more surprising, as the complete work does contain a chapter on "Numerical Integration and Summation." The probable explanation is that the separate publication of the first four chapters was an afterthought, and that it was then not possible to include the further chapter, which occurs rather later in the book.

Apart from this omission, the book will be found very useful. It is perhaps rather severely mathematical: there is very little, indeed, to suggest that interpolation is applicable to anything but mathematical tables. And the order adopted is perhaps not the best for the beginner. But the mathematician will find the book a useful compendium, and he will be grateful for the very full treatment of interpolation from unequal intervals. W. F. S.

Traité élémentaire des nombres de Bernoulli. Par NIELS NIELSEN. pp. x + 398. 50 fr. 1924. (Gauthier-Villars et Cie.)

An analytical treatise, produced in the twentieth century, which contains nothing so recondite as the sign of integration or (save in the preface) an infinite series or the elementary transcendental functions must surely be unique; and Prof. Nielsen's latest work satisfies this description. It is not very obvious what the answer should be to the question whether the Bernoullian numbers, the Bernoullian polynomials, the Eulerian numbers and the corresponding polynomials possess sufficient intrinsic interest to justify Prof. Nielsen's expenditure of energy in devoting to them a volume of four hundred pages—the size of his treatise on the *Cylinderfunktionen*; the present reviewer would probably, on consideration, be inclined to answer it in the affirmative. But a treatise such as this, in which the author is grappling with his subject with his strong right hand voluntarily tied behind his back, is one which seems to deserve admiration rather than sympathy.

The definition of the Bernoullian polynomials adopted in the book is based on the concept of a *harmonic sequence* of polynomials, namely a sequence $(f_n(x))$, such that $f_n(x)$ is of degree n in x, and such that $f_n'(x)=f_{n-1}(x)$. The Bernoullian polynomials $B_n(x)$ are defined as the harmonic sequence which satisfies the difference equation

$$B_n(x)-B_n(x-1)=x^{n-1}/(n-1)!,$$

and the Bernoullian numbers then appear naturally in connexion with the coefficients in the polynomials.

A similar procedure yields the Eulerian polynomials and the Eulerian numbers. This brings us to the end of Chapter III. The following nine chapters are devoted to what may broadly be described as recurrence formulae; and it is in reading these chapters that one cannot help feeling how immensely the book would have been improved by a judicious use of the transcendental, based on the fact that $B_n(x)$ is the coefficient of a^n in the expansion of

$$ae^{ax}/(1-e^{-a}).$$

The reader would then have been able to perceive the reason why the various formulae are inevitably true, and would have been satisfied that the author's results are exhaustive, instead of being reduced to admire the ingenuity with which the author has produced results from apparently nowhere. And it would have been quite possible for the author to have carried out such a scheme without for one moment abandoning his thesis that all the elementary properties of the Bernoullian numbers are obtainable by elementary methods.

Chapter XVI. deals with the representation of sums of powers by means of Bernoullian polynomials, and kindred topics; with this exception, Chapters XIII. to XX. are devoted to the remarkable theorems of Clausen and von Staudt, congruences and quadratic residues; for such investigations "elementary" methods are usually, though not invariably, the more appropriate, and consequently these chapters are by far the most interesting part of the book.

After remarking that readers of Prof. Nielsen's earlier works will find that this volume is well up to the exacting standards of completeness which the author has habitually set himself, and, in particular, that the author has pointed out numerous instances of mathematicians who are usually credited with discovering theorems on Bernoullian numbers having been anticipated by earlier writers—though the book is marred by the absence of an index, after the exasperating habit of French publishers—may we conclude by hoping that we shall soon see from the same pen an "Advanced Treatise" on Bernoullian numbers to deal with their transcendental aspects, such as their occurrence in the asymptotic expansions of various classes of integral functions? G. N. W.

The Foundations of Einstein's Theory of Gravitation. By E. FREUNDLICH. Translated by H. L. BROSE. 2nd ed., revised and enlarged. Pp. xvi + 140. 6s. net. 1924. (Methuen.)

The Theory of Relativity. Three Lectures for Chemists. By E. FREUNDLICH. Translated by H. L. BROSE. Pp. xii + 98. 5s. net. 1924. (Methuen.)

It is rather difficult to understand for what class of readers Dr. Freundlich's *Foundations of Einstein's Theory of Gravitation* is intended. It is not a systematic treatise suitable for students, and yet it can hardly be intended for the general public, as it seems to assume some familiarity with physics and mathematics. However, the fact that the book has reached a second edition shows that it has attracted a large number of those interested in relativity. It has a preface by Einstein, who says " I have gained the impression in perusing these pages that the author has succeeded in rendering the fundamental ideas of the theory accessible to all who are to some extent conversant with the methods of reasoning of the exact sciences." Prof. H. H. Turner contributes a short introduction, and the translator has added two essays, one of which contains an account of Dr. Freundlich's original work in connection with Einstein's predicted spectral shift. Three methods were employed. One compared statistically the spectra of a great many stars. Another compared the spectrum of a nebula with that of a star embedded in it. The third compared the fixed calcium lines in the spectrum of a double star with the other lines, which have periodic motions. The results were favourable to Einstein's theory, but they were far too rough to be conclusive. Towards the end of 1923 St. John, of the observatory on Mount Wilson in the United States, published observations that appear to agree very closely with Einstein's calculations. In former years St. John had believed that Einstein was wrong. This important matter is not mentioned in the book; probably the author finished his revision before the publication of the latest results.

The other book by Dr. Freundlich is much shorter and more elementary. A great deal of it should be intelligible to anyone. In spite of its title, it has no reference to chemistry. The account of the experimental work is not quite up to date. Viscount Haldane has written an appreciative preface, laying stress upon the finiteness of the velocity of light and the relation between mass and energy. H. T. H. PIAGGIO.

Éléments de Calcul Différentiel et Intégral. (Edition Revue.) Par W. A. GRANVILLE, en collaboration pour l'édition avec P. F. SMITH. Traduit de l'anglais par A.-A.-M. SALLIN. Differential Calculus, pp. 1-320; Integral, including ordinary differential equations, 321-533. Index, 535-537. Table of Contents, 539-548. 30 fr. 1924. (Vuibert, Paris.)

This excellent book covers the ground required by most practical students, and is intended also to prepare the mathematical student for more advanced work. The exposition is clear, the volume is illustrated by a great number of excellent diagrams, and is a storehouse of examples, pure and applied, which should be of great use to teachers as well as students.

It is based on the method of limits, but there is a very instructive chapter on differentials, a differential of a function of x being defined as the part of the complete difference of which Δx is a factor, higher powers of Δx being omitted. It is, therefore, not necessarily infinitesimal, but the points to which it applies (in a graph of the function) lie on the tangent to the curve, not on the curve itself. This is a method of treatment which has been adopted by several recent authors.

There are chapters dealing with change of variable, functions of several independent variables, series, curve tracing, and solid geometry.

The Integral Calculus covers the ordinary ground with applications up to moments of inertia. It does not deal with special definite integrals which can be evaluated only for special limits. The differential equations considered are the various types of linear equations, including those of the n^{th} order with constant coefficients. The final chapter deals with approximate integration and the use of a planimeter. A. LODGE.

Chance and Error. By Marsh Hopkins, B.A.Sc., M.E.I.C., D.L.S. Pp. 223. 7s. 6d. net. 1923. (Kegan Paul, Trench, Trubner & Co.)

"The Theory of Evolution" is the sub-title of this book; and that is enough to arouse very high hopes, for was there not a crying need for a popular book on Probability with especial reference to Evolution? If there was, then the need is still crying, for this book cannot be said to have in any sense supplied it.

Dealing with a subject which has evidently for the author, as for the reviewer, a fascinating interest, Dr. Hopkins at once enlists our sympathy by the confession that "the subject took such a hold of me that for a number of years I constantly worked at it in my sleep." "It was written with the object of extending the usefulness of this very important subject to those whose knowledge of mathematics is limited." Obviously a charming book—until you come to read it; and then the first page shatters the illusion. "Omnium consensu capax imperii," said Tacitus of Galba, "nisi imperasset."

The language is an odd mixture of the excessively formal and the excessively colloquial, generally in the wrong order. A chatty explanation to enlighten the beginner, leading to a more professionally worded statement, may offend the philosopher; but it does help the ordinary learner along. Dr. Hopkins' method is very different. Starting with a most repellent array of 19 formidable definitions, such as "The weight of any set of conditions is equal to the number of times that an observation made under that set of conditions must be taken to produce the same number of chances of a pair of contradictories as a single observation made under a set of conditions whose weight is 1," he follows it up with no discussion of their meaning but with such hail-fellow-well-met, slap-him-on-the-back explanations as this:

"Anything that travels far without supplies must make a lot of motion without expending much energy. . . . A force represented by the radius and acting on the centre of a circle may revolve around that centre without doing any work. And at a terrific rate. . . . Space is full of pulsations. . . . They travel far but do no work, or very little, because it is a constant give and take. . . . This is why wireless telegraphy travels so far, and so fast. Little time for motion, only force."

Most of us welcome informal language like Professor Silvanus Thompson's when it serves the purpose of making the principles clear; but can this be said of Dr. Hopkins' boisterous imitation of Mr. Jingle? Or is it intended as a counterpoise to the funereal solemnity of "All lines tending to contradictories are in the same plane, called the real plane"?

Purists who object even to the old-fashioned "$\underset{n=\infty}{\mathrm{Lt}}$" will be shocked by

$$\text{“}(1+0)^{\frac{1}{0}} = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}},\ \text{etc.,” and “}\underline{|s}\ \text{if}\ s < 1,\text{”}$$

and "The error in a plane target is equal to the error in a line multiplied by a right angle." But boorish statement of a truth may be forgiven when it can be understood; our author's chief fault is not seen in his use of intelligible slang like this but in his utterly incomprehensible "7·6 ways of making a different combination," "Any succession of a contradictory can be represented in magnitude and direction by a straight line," "We must divide 10 by the sum of the stakes to get the yes and no," and "e is the total number of chances of a combination from anything when all the elements are very small."

Reading the book soon degenerates (for those whose knowledge of mathematics is limited) into a competition to deduce from the answer the meaning of the question; and to most of the proofs the Corollary might well be attached, "In the same way it may be proved that all squares are circular. And enormously triangular."

At the end of the book the reader realises that he has not yet reached the part of it dealing with Evolution. He will, though sadly ruffled, have retained his affection for the author and his admiration for his obvious sincerity and enthusiasm, unless, of course, he is of an unusually cynical disposition. In that case he may possibly, after identifying the part at which the author worked in his sleep, ask where to find the part of it at which he worked when awake.

W. Hope-Jones.

Elements of the Theory of Infinite Processes. By L. L. SMAIL. Pp. viii, 340. 17s. 6d. 1923. (McGraw-Hill, New York and London.)

The reader is supposed to understand limits with regard to a continuous variable, and to know the meaning of continuity. Five pages deal with orders of infinity and four with infinite integrals, and there are chapters on "Elementary Functions" (16 pp.), "Trigonometric Developments" (19 pp.), and "Higher Transcendents" (22 pp.), but really this is a book on series and other sequences, with a range wide enough to include sections on Dirichlet series, asymptotic series, Borel and Cesàro summation, and infinite determinants. Since the pages are small and the print is large, the treatment is in some places slight to the verge of uselessness—who can benefit by the statement that every function which satisfies "certain conditions called Dirichlet's conditions" can be represented by a Fourier series, unless there is at least a hint that the unexplained conditions admit all everyday functions?—but each chapter ends with references to a few standard works. There is a small collection of examples at the end of the book.

The student must learn comparison tests and ratio tests, and it is with series, not with sequences of any other kind, that infinite integrals have analogies. Moreover, many experienced teachers affirm that the ideas of convergence are acquired most readily if attention is restricted to series. But it is of the utmost importance, educationally, to distinguish between the ideas which attach to the notion of sequence and the technical rules for dealing with sequences of special kinds, and against the weight of the authorities on whom I am most willing to rely, I have always resented the emphasis laid on the theory of series in the early training of the mathematician. To take one striking example, it is one thing to derive criteria of convergence for products from those for series, but it is another to *define* uniform convergence of the product $\Pi\, b_n(z)$ with Chrystal by a condition involving the difference

$$\prod_{m}^{\infty} b_n(z) - 1,$$

or with Goursat as *meaning* uniform convergence of an equivalent series. It will be realised then that it was with excitement that I read the title of Dr. Smail's book, and with pleasure that I found him to be setting to work along interesting lines, dealing first with aggregates and then with sequences and not reaching series until the seventh of his short chapters. But the mistrust aroused when in § 3 obscure language was found to conceal a sheer blunder * has not been dispelled, and I have come to the end somewhat disappointed.

In the first place, although definitions with regard to products, continued fractions, and determinants are framed with direct reference to the sequences involved, the hope of an outlook unusually wide is fulfilled only to a moderate extent. For example, uniform convergence is defined first for a series, and it is in relation to series that continuity, integrability, and Tannery's theorem are discussed; the definition of uniform convergence for a sequence is given subsequently, and we are told that by a slight modification of the former proof † the continuity of the limit can be demonstrated, but this is not the same as insisting that it is to the sequence as a function of two variables that the idea belongs; Tannery's theorem is not adapted even to products, and therefore is not available for the discussion of sin x, to which it is peculiarly suited.

It is hard to know what is taken for granted with regard to the exponential and logarithmic functions; if the treatment of these was more than incidental it would hardly bear inspection. One should perhaps examine such a book as this on the unpalatable assumption that it is possible to be over-scrupulous, but at least it must be said that in matters of detail the author is neither

* If E_1 and E_2 are the classes of real numbers separated by a rational number k, the A_1 and A_2 of the argument are the classes of rational numbers less than and greater than k, and k does not belong to either class. The pretended deduction that there cannot be such a number is therefore invalid. It is easy to correct the proof, but that is hardly the reader's business.

† Unfortunately for the beginner a misprint refers him not to any proof at all but to the definition of conditional convergence!

reliable nor careful. For example, the discussion of the product for $\sin x$ is about as bad as an exposition of a sound method could be : there is no attempt to help the learner to appreciate the points of the argument, there are six mistakes and four misprints, and although the proof uses inequalities which assume that the variable is real, ix is immediately substituted for x in the result.

Throughout the book misprints are too common ; in particular, $|z$ or $z|$ has perpetually to do duty for $|z|$. Several French accents are wrong, and even in page-headings I have noticed mistakes. The breaking of formulae at the turn of pp. 229 and 315 should have been somehow avoided. While $n!$ looks innocent, $2!$ and $(n-1)!$ are intolerable. To add to the reader's difficulties, the sentences are not all well composed. For example, a proof on p. 81 concludes : "The series $\Sigma 1/n(n+3)$ has $Lb_{n+1}/b_n=1$, so the ratio test fails, but by comparing with $\Sigma 1/n^2$, we have $1/n(n+3)<1/n^2$, hence the series is convergent." And on p. 302, the sentence "It suffices to prove the convergence of the series (b) to shew that the series $\Sigma' 1/w^3$ is convergent" means "*In order* to prove the convergence of (b) it is sufficient to shew that $\Sigma' 1/w^3$ is convergent."

To sum up, I do not know a book in English which would suggest to a competent teacher a more stimulating programme for an introductory course, but the teacher must be on his guard against mistakes, and unless he has confidence in his own judgment he must have access to some of the trustworthy treatises to which references are given. For students, the book can be judged most favourably as an introductory text-book on series ; as such it is well arranged and comprehensive, and to those who can afford the price it can be recommended, with only the warnings that the excursions into analysis must be ignored and the style of the proofs is not always above reproach.

Dr. Smail adopts from Pierpont the practice of dropping the connecting words from such a sentence as *If* $\epsilon>0$ *there is a* ν *such that* $|U_n-U|<\epsilon$ *whenever* $n>\nu$, and writing simply the row of symbols

$$\epsilon>0, \quad \nu, \quad |U_n-U|<\epsilon, \quad n>\nu.$$

If abbreviation is wanted, would it not be better to adopt the notation of *Principia Mathematica* and to write

$$\epsilon>0 \supset : (\exists \nu) . n>\nu \supset |U_n-U|<\epsilon ?$$

I recommend readers of the *Gazette* who have not made the experiment to express in this style the condition that a sequence involving a variable x is convergent for any value of x in a specified range ; they will find that the distinction between uniform and non-uniform convergence leaps to the eye.

E. H. N.

A Course of Experimental Mechanics. By H. J. E. BAILEY. Pp. xv +223. n.p. 1924. (Chapman & Hall.)

The revolution that has taken place in the present century in the teaching of Mechanics is well illustrated by this book. Now-a-days the danger is to give too much instead of too little attention to the experimental side. The teacher has not time to demonstrate every particular case of a Fundamental Principle by a separate experiment, and if he makes a judicious choice he will get better results than if he attempted a more detailed method. The performance of an experiment in class by the master has for certain types of boy small educational value, and the reviewer has much sympathy with these types. The slacker enjoys it, and the boy who really likes Mechanics is apt to be bored because he wants to do the thing himself. For the boy it is different, he has to get some experimental dexterity, and can do so only by going through a graduated course such as is provided by Mr. Bailey's book.

The experiments on Statics include only two on the Strength of Materials—Young's Modulus and the Breaking Stress of a Wire. To these might have been added simple cases of the Modulus of Torsion and the Deflection of a Beam. Fletcher's Trolley is used for experiments on Uniform Acceleration, Conservation of Momentum and Conservation of Energy. Goodwill's Vector Balance is also used for experiments on Collision. A very full discussion is given of Capt. Kater's Pendulum.

There are two Appendices; the first collects together Definitions, Formulae and Statements of the Main Principles and the second gives the Mathematical Proofs required in some of the experiments. The proof that the Mean-Position Velocity of a Simple Pendulum is proportional to the Amplitude depends on the assumption that the motion is a Simple Harmonic Motion. A better and more instructive proof is to show by the Conservation of Energy that the Mean-Position Velocity is accurately proportional to the chord of half the arc, and therefore approximately proportional to the Amplitude when the arc is small.

In some of the experiments there is an over elaboration of details, but the general scheme for setting out the results and the suggestions are excellent. The book can be thoroughly recommended. R. M. M.

Vector Analysis. By C. Runge. Translated by H. Levy. Pp. viii, 226. 9s. 1923. (Methuen.)

From meeting the demand for popular expositions of the theory of relativity, Messrs. Methuen have been led on to the publication of a serious mathematical work, to which only the thickness of the paper and an occasional eccentricity in the spacing of a formula give an unprofessional touch. The book was published in German in 1919. Prof. Runge has in preparation an account of vector analysis in the eventful fourfold of the theory of relativity, and this book on three-dimensional vector analysis was designed as a preliminary volume. The methods of development have been chosen with reference to what is to follow. The ground covered is to a great extent common to books on the subject, and the work falls naturally into three divisions.

I. Vector algebra, including the use of a vector frame in which the vectors of reference are not assumed to be unit vectors or to be mutually orthogonal. The reader will be struck by the discrimination between the external product and the vectorial product of two vectors; these have the same numerical measure, but the first, which reflects Grassmann's share in the invention of the subject, is a two dimensional entity in a definite vecplane, and the second, which reflects Hamilton's share, is an ordinary vector perpendicular to this vecplane. The scalar product used is Grassmann's, the negative of Hamilton's; it is defined first as a volume. The notation is that of Gibbs, and the dot which should indicate a scalar product is missing often enough to emphasise this danger in the notation.

II. Differentiation and integration, including discussions of twisted curves, of relations between line, surface, and volume integrals, and of the operator ∇. The systematic arrangement of the theorems on integration that involve ∇ is very attractive, and the author does not forget to prove that ∇ is independent of the frame of reference.

III. Tensors, introduced by a transformation of vecspace in which corresponding vectors have identical coefficients referred to different frames. Symmetric, antisymmetric (they are called *asymmetric* here, but surely there is no excuse for appropriating this name for a special kind of unsymmetric tensor), and rotational tensors are described, the general tensor is analysed, and there are sections on tensor fields and tensor integrals.

There are no excursions into elementary geometry and few into elementary mechanics, there is no mention of rotors or moments, and there is only one exercise set to the reader, but as an exposition of the pure theory of vectors, the work is well worthy of the author's high reputation. The English student has cause to be very grateful for the translation.

Bell's Card of Logarithms and Science Tables. $8\frac{1}{4}$ in. × 10 in. 2s. per dozen. (Bell & Sons.)

On one side, four-figure logarithms, without split difference-columns. On the other, circular measure and sines and tangents at intervals of one degree, symbols and atomic weights of forty elements, a dozen mathematical and physical constants, a dozen relations between different units of measurement, and half a dozen formulae in mensuration. The card is stout and the printing bold. E. H. N.

THE LIBRARY.

160 CASTLE HILL, READING.

ACKNOWLEDGMENTS are due to Col. R. L. Hippisley, Mr. T. W. Hope and others, but for reasons of space details must be held over until the next number of the *Gazette*.

In future, entries will include whatever information the Librarian has as to the edition of each book. For this purpose brackets of the form { } will be appropriated. The absence of these brackets will shew that there is no indication that the book is not in its first edition—we have books, from America and India, with the proud words *First edition* on the title page, but the temptation to distinguish these by {1} will be resisted. When an edition is stated, or is otherwise known, to be a reprint, as much will be made clear; the date column will, of course, contain the date of the actual copy, but whenever possible the date of the original will be given also; for example, we may have either {2 (1913) rep.} or {4, *i.e.* 2 (1913) rep.} according to whether the book though in fact a reprint is described as such or as another edition. Sometimes an edition is said to be ' new ' or ' revised ' but is not numbered; empty brackets will shew that this is the case and that the compiler has no supplementary knowledge. The Librarian will be very grateful to anyone who sends him details to fill in a blank of this kind, or indeed who corrects his lists in any way; information will not come too late to be of service to the Association, for it will be incorporated in due course into a Catalogue.

THE GODFREY MEMORIAL.

THE collection of books which, as was said in the July number of the *Gazette*, Mrs. Godfrey has given to the Association, is now in the Library. Each volume bears the inscription reproduced here. In her last letter on the subject, Mrs. Godfrey writes: " I feel so very glad that my husband's books have gone somewhere where they really may be of use and interest. He used them himself very thoroughly." Mr. Siddons writes: "This gift may possibly help to stimulate the use of the Library a good deal; that would be a great memorial." It is for members now to shew their appreciation, and they will see from the details which follow that the range of interests is wide.

LIBRARY OF THE
MATHEMATICAL
ASSOCIATION.

In lasting memory of
CHARLES GODFREY
1873 - 1924
an ardent reformer of
mathematical teaching
and a tireless worker
for the Association, his
collection of books on
mathematics and on the
teaching of mathematics
was given to the Library
by his wife.

1. The works in which Professor Godfrey collaborated:

With G. M. BELL	The Winchester Arithmetic - - - *Teachers' edition, interleaved with answers.*	1905
	Note-Book of Experimental Mathematics - -	1905
With E. A. PRICE	Arithmetic - - - - - - -	1915

With A. W. SIDDONS	Algebra for Beginners	1912
	Elementary Algebra (2 vols.) {I, 2; II, 1 (1913) rep.}	1920, 1917
	Exercises from *Elementary Algebra*	1920
	First Steps in Calculus (Chapters from *Elementary Algebra*, with exercises)	1914
	Four-Figure Tables	1913
	Geometry for Beginners	1909
	Elementary Geometry {3 (1912) rep.} *A reprint executed in India.*	1920
	Modern Geometry {1 (1908) rep.}	1920
	Solutions of the Exercises in *Modern Geometry*	1909
	Practical Geometry and Theoretical Geometry	1920
	Shorter Geometry {1 (1912) rep.}	1921
	Solid Geometry {1 (1911) rep.}	1920
	Shorter Geometry and Solid Geometry *The two books reprinted from the original editions* (1912, 1911) *and bound together.*	1917

The following keys:

C. L. BEAVEN	Solutions of the Exercises in Godfrey and Siddons's *Solid Geometry*	1912
E. A. PRICE	Solutions of the Exercises in Godfrey and Siddons' *Elementary Geometry*	1904
	Solutions of the Exercises in Godfrey and Siddons's *Shorter Geometry*	1914

A translation:

F. HOCEVAR	Solid Geometry. Translated from German and adapted by C. Godfrey and E. A. Price *The original is the second part of Hocevar's* Lehr- und Übungsbuch der Geometrie für Untergymnasien; *Prof. Godfrey's copies of three editions of this book form part of the memorial.*	1903

2. A number of publications of the International Commission on the Teaching of Mathematics (1908-1920), of which Prof. Godfrey was a member; these can be identified from the numbers attached to them in the official list:

Central Committee—	Nos, 7, 9-11.
National Sub-Commissions—	
Germany	Nos. 28, 33, 36-37, 49, 63-64.
Argentine	No. 65 (complete).
Australia	Nos. 66-71 (complete in one volume).
Austria	No. 83.
United States	Nos. 111, 114-120, 122-125.
France	Nos. 153-164 (complete in one volume).
British Isles	Nos. 195-228 (complete in two volumes, bound).
Japan	Nos. 240-255 (complete in two volumes, bound).

A number of A.I.G.T. *General Reports*, a long run of the *Gazette*, and copies of many of the Reports issued by the A.I.G.T. and the M.A.

3. The following books:

P. APPELL	Traité de Mécanique Rationnelle. I	1893
W. W. R. BALL	History of Mathematics {4}	1908
	History of the Study of Mathematics at Cambridge	1889
	Mathematical Recreations and Problems {3}	1896
A. B. BASSET	Elementary Hydrodynamics and Sound	1890
J. BECQUEREL	Relativité et Gravitation	1922
W. H. BESANT	Treatise on Dynamics {2}	1893
	Treatise on Hydromechanics. I: Hydrostatics {5}	1891
J. BOLYAI	Science Absolute of Space. Trans. from Latin with an introduction by G. B. Halsted {4}	1896
R. BONOLA	Non-Euclidean Geometry. Trans. from Italian by H. S. Carslaw	1912
G. BOOLE	Calculus of Finite Differences *Repairing a loss recorded last December.*	1860
S. BRODETSKY	Nomography	1920
T. J. I'A. BROMWICH	Quadratic Forms (Cambridge Tracts 3)	1906
D. BRUNT	Combination of Observations	1917
F. CAJORI	History of Elementary Mathematics	1896
W. T. CAMPBELL	Observational Geometry	1899
G. CANTOR	Transfinite Numbers. Trans. from German and annotated by P. E. B. Jourdain	1915
G. A. CARSE and G. SHEARER	Fourier's Analysis for the Mathematical Laboratory (Edinburgh Tracts 4)	1915

H. S. CARSLAW	Introduction to Infinitesimal Calculus {2} - - -	1912
J. CASEY	Sequel to Euclid {5} - - - - - - -	1888
G. CHRYSTAL	Algebra (2 vols.) {I, 3; II, 1} - - - - -	1893, 1889
W. K. CLIFFORD	Elements of Dynamic. Part I - - - - - - *Will any member give the concluding Part?*	1878
L. CREMONA	Projective Geometry. Trans. from Italian by C. Leudesdorf. {2} - - - - - - - - - - -	1893
E. CUNNINGHAM	Relativity, Electron Theory, and Gravitation {2} - -	1921
R. DEDEKIND	Essays on the Theory of Numbers. Trans. from German by W. W. Beman {1 (1901) rep.} - - - - -	1909
A. DE MORGAN	Study and Difficulties of Mathematics. Ed. by T. J. McCormack - - - - - - - - - - - *This essay originally formed part of the* Library of Useful Knowledge.	1898
A. C. DIXON	Elementary Properties of Elliptic Functions - - *This copy is a curiosity, being cased upside-down.*	1894
C. L DODGSON	Euclid and his Modern Rivals {2} - - - - - *This edition has a criticism of Henrici which was not in the first edition.*	1885
C. V. DURELL	Plane Geometry for Advanced Students. II - - - -	1910
A. S. EDDINGTON	Report on the Relativity Theory of Gravitation - - -	1918
	Space Time and Gravitation - - - - - - -	1920
A. EINSTEIN	Relativity. A popular exposition. Trans. from German by R. W. Lawson. {2} - - - - - - - -	1920
K. FINK	Brief History of Mathematics. Trans. from German by W. W. Beman and D. E. Smith - - - - -	1900
W. B. FRANKLAND	First Book of Euclid's Elements with a Commentary based upon that of Proclus - - - - - -	1905
W. S. FRANKLIN	Bill's School and mine - - - - - - - - -	1913
C. DE FREYCINET	De l'Expérience en Géométrie - - - - - -	1903
P. FROST	Curve Tracing {2} - - - - - - - - - -	1892
D. GIBB	Interpolation and Numerical Integration (Edinburgh Tracts 2)	1915
G. A. GIBSON	Introduction to Calculus - - - - - - -	1904
E. GOURSAT	Cours d'Analyse Mathématique (3 vols.) {3} - 1917, 1818,	1923
A. G. GREENHILL	Differential and Integral Calculus {2} - - - -	1891
	Notes on Dynamics {2} - - - - - - - - -	1908
	Hydrostatics - - - - - - - - - - - -	1894
G. B. HALSTED	Rational Geometry - - - - - - - - -	1904
J. HARKNESS and F. MORLEY	Introduction to the Theory of Analytic Functions - -	1898
A. HARNACK	Calculus. Trans. from German by G. L. Cathcart - -	1891
R. S. HEATH	Geometrical Optics - - - - - - - - -	1887
H. HERTZ	Electric Waves. Trans. from German by D. E. Jones -	1893
	Miscellaneous Papers. Trans. from German by D. E. Jones and G. A. Schott - - - - - - - -	1896
E. W. HOBSON	Plane Trigonometry - - - - - - - - -	1891
A. HÖFLER	Didaktik des Mathematischen Unterrichts - - - -	1910
C. S. JACKSON	Examples in Calculus - - - - - - - - -	1921
J. H. JEANS	Theoretical Mechanics - - - - - - - - -	1907
LORD KELVIN and P. G. TAIT	Natural Philosophy (2 vols.) {I, 1 (1879) ster.; II, 2} -	1896, 1895
G. KIRCHHOFF	Mathematische Physik. IV: Theorie der Wärme - -	1894
C. F. KLEIN	Nicht-Euklidische Geometrie (2 vols.) {1 (1890) rep.} -	1893
C. A. LAISANT	La Mathématique: Philosophie—Enseignement - -	1898
	Thresholds of Science: Mathematics - - - - - *This is a translation of* L'Initiation Mathématique *already in the Library. Far from adapting the work to English readers, the anonymous translator asserts that Pythagoras' Theorem is called the Asses' Bridge; the odd fact that the nickname is used of different propositions in different countries hardly makes the statement accurate* in English.	1913
H. LAMB	Hydrodynamics {2} - - - - - - - - - - *The earlier work* (1879) *bore a different title, but in subsequent editions that of* 1895 *is called the second.*	1895
	Infinitesimal Calculus - - - - - - - - -	1897
J. G. LEATHEM	Volume and Surface Integrals used in Physics (Cambridge Tracts 1)	1905

G. LECHALAS	Introduction à la Géométrie Générale - - - - -	1904
W. LIETZMANN	Der Pythagoreische Lehrsatz - - - - -	1912
E. S. LOOMIS	Original Investigation - - - - - - -	1901
G. LORIA	Vorlesungen über Darstellende Geometrie. Trans. from Italian manuscript into German. I - - - - -	1907
A. E. H. LOVE	Theoretical Mechanics - - - - - - - - -	1897
E. LUCAS	L'Arithmétique Amusante - - - - - - -	1895
J. MCDOWELL	Exercises on Euclid and in Modern Geometry - -	1863
E. MACH	Science of Mechanics. Trans. from German by T. J. McCormack - - - - - - - - - -	1893
H. P. MANNING	Non-Euclidean Geometry - - - - - -	1901
G. B. MATHEWS	Algebraic Equations - - (Cambridge Tracts 6)	1907
J. C. MAXWELL	Electricity and Magnetism (2 vols.) {3} - - - - - *See also* J. J. Thomson.	1892
	Matter and Motion - - - - - - - - - -	1882
G. MONGE	Géométrie Descriptive {5} - - - - - - - -	1827
I. NEWTON	Universal Arithmetick. Cunn's edition {2} - - - -	1728
J. PERRY	Elementary Practical Mathematics {1 (1913) rep.} - -	1920
	Practical Mathematics: Summary of Six Lectures to Working Men - - - - - - - - - -	1899
C. E. PICARD	Développement de l'Analyse et ses Rapports avec diverses Sciences - - - - - - - - - -	1905
	Traité d' Analyse (3 vols.) - - -	1891, 1893, 1896
M. PLANCK	Origin and Development of Quantum Theory - - *Nobel Prize Address* 1920, *translated.*	1922
T. PRESTON	Theory of Light {2} - - - - - - - -	1895
LORD RAYLEIGH	Theory of Sound (2 vols.) {2} - - - - - -	1894, 1896
R. REIFF	Theorie Molekular-Elektrischer Vorgänge - - -	1896
K. T. REYE	Geometry of Position. Trans. from German by T. F. Holgate. Part I - - - - - - - - -	1898
E. J. ROUTH	Analytical Statics (2 vols.) - - - - - -	1891, 1892
	Rigid Dynamics (2 vols.) {5} - - - - - -	1891, 1892
T. SUNDARA ROW	Geometrical Exercises in Paper Folding - - - - *The original Madras edition.*	1893
C. D. T. RUNGE	Graphical Methods - - - - - - - - -	1912
T. R. RUNNING	Empirical Formulas - - - - - - - - -	1917
B. A. W. RUSSELL	Foundations of Geometry - - - - - - -	1897
	Principles of Mathematics. I - - - - - - -	1903
G. SALMON	Analytic Geometry of Three Dimensions {4} - - - - *The last edition seen by the author.*	1882
M. SCHLICK	Space and Time in Contemporary Physics. Trans. from German by H. L. Brose - - - - - - - -	1920
KASPAR SCHOTT	Cursus Mathematicus, sive Absoluta omnium mathematicarum disciplinarum Encyclopædia, in Libros XXVIII digesta, eoque ordine disposita, ut quivis, vel mediocri praeditus ingenio, totam Mathesin à primis fundamentis proprio Marte addiscere possit. Opus desideratum diu, promissum à multis, à non paucis tentatum, à nullo numeris omnibus absolutum { }	1699
H. SCHUBERT	Mathematical Essays and Recreations. Trans. from German by T. J. McCormack - - - - - - -	1898
L. SILBERSTEIN	Theory of Relativity - - - - - - - -	1914
	Vectorial Mechanics - - - - - - - - -	1913
D. E. SMITH	Teaching of Arithmetic {4} - - - - - - - -	1911
	Teaching of Elementary Mathematics - - - - -	1900
J. SMITH	Quadrature of the Circle - - - - - - - - *Readers of the* Budget of Paradoxes *will recognise "the ablest head at unreasoning, and the greatest hand at writing it, of all who have tried in our day to attach their names to an error." This is the smallest of his books on the subject.*	1865
J. STEWART	Newton's *Quadrature of Curves* and *Analysis by Equations of an infinite Number of Terms*, translated (from Latin) with a large Commentary - - - - - - -	1745
E. STONE	The Method of Fluxions both Direct and Inverse. The Former being a Translation from the Celebrated Marquis De L'Hospital's *Analyse des Infiniment Petits*: and the Latter supply'd by the Translator -	1730
P. G. TAIT	Newton's Laws of Motion - - - - - - -	1899
	Quaternions {2} - - - - - - - -	1873

P. G. TAIT and W. J. STEELE	Dynamics of a Particle {6}	1889
J. TANNERY	Arithmétique	1894
	Notions de Mathématiques {2, *i.e* 1 (1903) rep.}	
C. TAYLOR	Ancient and Modern Geometry of Conics	1881
J. J. THOMSON	Recent Researches in Electricity and Magnetism. A Sequel to Clerk-Maxwell's Treatise	1893
I. TODHUNTER	History of the Progress of the Calculus of Variations during the Nineteenth Century	1861
R. TOWNSEND	Modern Geometry (2 vols.)	1863, 1865
J. VENN	Logic of Chance {3}	1888
H. WEBER	Algèbre Supérieure. Trans. from German into French by J. Griess	1898
L. D. WELD	Errors and Least Squares	1916
H. WEYL	Raum, Zeit, Materie {4}	1921
A. N. WHITEHEAD	Axioms of Descriptive Geometry (Cambridge Tracts 5)	1907
	Axioms of Projective Geometry (Cambridge Tracts 4)	1906
	Principle of Relativity	1922
E. T. WHITTAKER	Modern Analysis	1902
	Theory of Optical Instruments (Cambridge Tracts 7)	1907
B. WILLIAMSON	Differential Calculus {6}	1886
	Integral Calculus {6}	1891
J. E. WRIGHT	Invariants of Quadratic Differential Forms (Cambridge Tracts 9)	1908
J. W. A. YOUNG	Teaching of Mathematics in the Elementary and the Secondary School	1907
	Teaching of Mathematics in Higher Schools of Prussia	1900
G. U. YULE	Introduction to Theory of Statistics {5}	1919

ANONYMOUS:

Géométrie et Travaux Manuels. Cours élémentaire {2} - 1898

Géométrie : Dessin et Travaux Manuels. Cours moyen - 1897
Two volumes of : Cours des Ecoles Primaires Elémentaires publié sous la direction de E. Cazes.

A Mathematical Miscellany in four Parts. By a Lover of the Mathematicks. {3} - 1751
Published at London : M. Cooper. There were editions at Dublin 1730, 1735, 1770.

Mémento de Géométrie par un Groupe d'Instituteurs - (1899
Published at Paris : Delagrave.

REPORTS:

British Association:

Discussion on the Teaching of Mathematics - 1901
Discussion on the Teaching of Elementary Mechanics - 1906
The discussion took place at Johannesburg in 1905.

International Congress of Mathematicians:
Proceedings of the Fifth Congress. Cambridge, 1912 (2 vols.) - 1913

EXAMINATION QUESTIONS:

A Collection for the use of the Cadets of H.M.S. "Britannia" - 1890

Winchester College Examples - 1900

4. School-books and text-books by:

P. Abbott, C. H. Allcock, C. E. Ashford, E. H. Askwith;

W. M. Baker and A. A. Bourne, E. Bardey (2), S. Barnard and J. M. Child (4. 5 vols.), D. Behrendsen and E. Götting, W. W. Beman and D. E. Smith (2), P. Bert, W. H. Besant (3), W. G. Borchardt (2), E. Borel (2 vols.), H. Bos, C. Bourlet, J. G. Bradshaw, D. Brent, W. Briggs and G. H. Bryan, E. J. Brooksmith;

G. E. St. L. Carson and D. E. Smith (2, 4 vols.), F. Castle (3), G. Chrystal, A. Consterdine and A. Barnes, A. G. Cracknell;

A. Dakin, C. Davison (2), A. De Morgan (2), W. J. Dobbs, C. V. Durell;

J. Edwards, W. D. Eggar (3), W. T. A. Emtage;

R. C. Fawdry, R. C. Fawdry and C. V. Durell, D. F. Ferguson and H. E. Piggott, G. E Fisher and I. J. Schwatt, W. C. Fletcher;

J. H. Grace and F. Rosenberg, J. Greaves;

J. Hadamard (2 vols.), H. S. Hall (2, 3 vols.), H. S. Hall and S. R. Knight (2), H. S. Hall and F. H. Stevens (5, 6 vols.), J. G. Hamilton and F. Kettle, J. Harrison, R. Haussner R. B. Hayward, J. Henrici and P. Treutlein, O. M. F. E. Henrici, O. M. F. E. Henrici and G. C. Turner, W. M. Hicks, G. A. Hill, F. Hocevar (5);

C. S. Jackson and W. M. Roberts, H. S. Jones (1, 2 vols.);

J. P. Kirkman and A. E. Field, P. Klauke;

R. Lachlan (2), R. Lachlan and W. C. Fletcher, E. M. Langley and W. S. Phillips, A. E. Layng, A. M. Legendre and A. Blanchet, O. Lesser (2, 3 vols.), R. Levett and C. Davison (2), P. Leyssenne, J. B. Lock (3), A. Lodge, S. L. Loney (2), A. E. H. Love ;

Frère G. M. (2), D. B. Mair (2), C. Méray, J. W. Mercer (2), W. P. Milne and G. J. B. Westcott, G. M. Minchin ;

R. Nettell and H. G. W. Hughes-Games, R. C. J. Nixon (2) ;

W. F. Osgood ;

C. Pendlebury (2), J. Perry, H. E. Piggott, A. J. Pressland (2), E. A. Price ;

G. Richardson and A. S. Ramsey, H. A. Roberts, E. Rouché and C. de Comberousse (2 vols.);

F. W. Sanderson, F. W. Sanderson and G. W. Brewster, F. M. Saxelby, F. M. Saxelby and C. H. Saxelby, A. Schülke, W. J. Schüller, A. Schultze and F. L. Sevenoak, K. Schwab (3 vols.) P. Scoones and L. Todd, A. W. Siddons and A. Vassall, C. Smith (3), C. Smith and S. Bryant, J. H. Smith, W. W. Speer (3);

H. M. Taylor, I. Todhunter (2), I. Todhunter and J. G. Leathem, I. Todhunter and S. L. Loney, P. Treutlein, C. O. Tuckey and W. A. Nayler, J. F. Twisden ;

P. W. Unwin, G. Veronese ;

J. Watson, G. A. Wentworth (2), G. A. Wentworth and G. A. Hill, R. Wormell.

---

265. F. W. Newman, brother of the Cardinal, went up to Worcester College when he was seventeen, and obtained a Double First in 1826. He is said to have been the first man who ever offered in the Schools the Higher Mathematics analytically treated. Only one of his examiners understood the subject ; and he reported to his colleagues answers so brilliant that, besides awarding a First Class, they presented the candidate with finely bound copies of *La Place* and *La Grange.*

266. I thought it proper to insert in this Introduction *A Compendium of Algebra*, whose Name I know ought not to scare the Reader, for 'tis only a Method of Reasoning by the help of the Letters of the Alphabet, representing the Quantities, whose Relations are consider'd ; and it is to the Mathematicks the same that Logic is to the ordinary Philosophy, and therefore has been called *Logistic* and it is become so common amongst us, because of its engaging Beauty, and vast Use in all parts of the Mathematicks, that even Ladies of the highest Quality have been induced to learn it ; the Dutchess of E—— has attain'd so great a Degree of Perfection, as well in Numbers as Geometry, that Persons who make the greatest Figure for Learning have earnestly sought for the Honour of her Conversation. An Instance so illustrious ought to banish all sorts of Diffidence, and excite those that love their Ease.—Preface to vol. i. of the above. Ozanam, *Cursus Mathematicus.* Pref. Vol. I. 1712.

267. One has heard of a mathematician who said he could return from seeing the Russian Ballet and reconstruct for his own satisfaction in algebra the patterns traced on the stage by the dancers.—*Manchester Guardian,* April, 1922.

268. Byron, in reference to his wife's love for mathematics, called her the " Princess of Parallelograms."

269. At a late, lonely, and humble supper, the little table at which I wrought theoretical and practical mathematics, the very small pile of books (but oh, how valued ! and then how really valuable !), the wretched light, the fireless grate, the damp, cold stone floor, the aching head, the swollen feet, the shivering frame, and that which enabled me to bear the whole—the determination to know something of the beautiful and astonishing universe.—William Pengelly, F.R.S.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
AT THE UNIVERSITY PRESS, GLASGOW.

BOOKS RECEIVED, CONTENTS OF JOURNALS, ETC.

October, 1924.

Traité élémentaire des Nombres de Bernoulli. By N. NIELSEN. Pp. x+398. 50 fr. 1924. (Gauthier-Villars.)

Eléments de Calcul Différential et Intégral. Par W. A. GRANVILLE. Revised by P. F. SMITH. French translation by A. A. M. SALLIN. Pp. vii+548. 30 fr. 1924. (Vuibert, Paris.)

Die Naturliche Geometrie. By J. HJELMSLEY. Hamburger Mathematische Einzelschriften. Heft 1. 1923.

Über Analysis Situs. By H. TIETZE. Pp. 32. Hamburger Mathematische Einzelschriften. Heft 2. 1923.

Vorlesungen die Grundzüge der Mathematischen Statistik. By C. V. L. CHARLIER. Pp. 127. (Scientia, Lund.)

Analytical Mechanics. By E. H. BARTON. 2nd Edit. Pp. xxi+593. 21s. net. 1924. (Longmans, Green.)

A Companion to Elementary School Mathematics. By F. C. BOON. Pp. 302. 14s. net. 1924. (Longmans.)

Petit Traité de Perspective. By R. BRICARD. Pp. 87. n.p. 1924. (Vuibert, Paris.)

Les Lieux Géométriques. By T. LEMOYNE. Pp. 146. 10 frs. 1924. (Vuibert, Paris.)

The Unification of Mathematical Notations in the Light of History. By F. CAJORI. Pp. 87-93. Reprint from *The Mathematics Teacher*, Feb. 1924.

Relativity for Physics Students. By G. B. JEFFERY. Pp. 150. 6s. net. 1924. (Methuen.)

College Algebra. By L. P. SICELOFF and D. E. SMITH. Pp. 258. $1.80. 1924. (Ginn.)

Rahn's Algebraic Symbols. By F. CAJORI. Pp. 65-71. Reprint A, *Math. Monthly.* Feb. 1924.

The Reorganization of Mathematics in Secondary Education. A Reprint by the National Committee on Mathematical Requirements. Pp. x+652. n.p. 1923. (The Mathematical Association of America, Inc.)

The Numeral-Words: their Origin, Meaning, History, and Lesson. By M. DE VILLIERS. Pp. 124. n.p. 1923. (Witherby, 326 High Holborn, W.C.)

Hydrodynamics. By H. LAMB. 5th Edit. Pp. xvi+687. 45s. net. 1924. (Cam. Univ. Press.)

Cambridge Readings in the Literature of Science. Arranged by W. C. D. and M. D. WHETHAM. Pp. x+275. 7s. 6d. net. 1924. (Cam. Univ. Press.)

Die Mathematische Methode: Logisch Erkenntnistheoretische Untersuchungen im Gebiete der Mathematik, Mechanik und Physik. By O. HÖLDER. Pp. x+563. $6.30. 1924. (Springer, Berlin.)

A Shorter School Geometry. Part I. By H. S. HALL and F. H. STEVENS. Pp. x+164+iv. 2s. 6d. 1924. (Macmillan.)

Exercises in Trigonometry. By E. R. PIGROME. Pp. 78. 1s. 6d. 1924. (Clarendon Press.)

A Preparatory Arithmetic. By C. PENDLEBURY. Pp. xiv+290+xlviii. 3s. 1924. (Bell & Sons.)

Revision Arithmetic and Mensuration. By T. THOMAS and J. J. P. KENT. 3rd Edit. Pp. 1-128. 2s. net. 1924. (Mills & Boon.)

The Mathematical Groundwork of Economics: An Introductory Treatise. By A. L. BOWLEY. Pp. viii+98. 7s. net. 1924. (Clarendon Press.)

Abhandlungen aus dem Mathematischen Seminars der Hamburgischen Universität. (Hamburg : im verlag des Math. Seminars.)

III. 2. May, 1924.

Zur additiven Primzahltheorie algebraischer Zahlkörper. I. Über die Darstellung totalpositiver Zahlen als Summe von totalpositiven Primzahlen im reell-quadratischen Zahlkörper. Pp. 109-163. H. RADEMACHER. *Eine topologische Kennzeichnung der Kreise auf der Kugel.* Pp. 164-166. W. BLASCHKE. *Über die Gruppen $A^aB^b=1$.* Pp. 167-169. O. SCHREIER. *Ein mechanisches System mit quasiergodischen Bahnen.* Pp. 170-175. E. ARTIN. *Über die Geometrie von Laguerre: Grundformeln der Flächentheorie.* Pp. 176-194. W. BLASCHKE.

Academia pro Interlingua. (Torino.)

April, 1924.

The American Mathematical Monthly. (Lancaster, Pa.)

March, 1924.

The Algebra of Correlation. Pp. 110-121. D. JACKSON. *Two Models in Statistical Mechanics* Pp. 121-126. A. J. LOTKA. *The Numbers of Representations of Integers of Certain Forms, $ax^2+by^2+cz^2$.* Pp. 126-131. E. T. BELL. *Conical Loci associated with the Motions of a Rigid Body about a Point.* Pp. 131-135. E. L. REES. *A Property of the Isogonal Centres of a Triangle.* P. 135. H. M. LUFKIN. *Circumscribed and Inscribed Tetrahedra.* Pp. 135-137. A. A. BENNETT. *On Algorithms for the Solution of the Linear Congruence.* Pp. 137-140. H. S. VANDIVER.

April, 1924.

The Present State of the Difference Calculus and the Prospect for the Future. Pp. 169-183. R. D. CARMICHAEL. *On Daniel Bernoulli's "Moral Expectation and on a new Conception of Expectation."* Pp. 183-190. C. JORDAN. *Simple Derivations of the Formulas for the Dispersion of a Statistical Series.* Pp. 190-196. *Some Limit Proofs in Solid Geometry.* Pp. 196-202. W. R. LONGLEY. *An Approximate Construction of the Side of a Regular Inscribed Nonagon.* P. 202. T. R. RUNNING.

May, 1924.

Determinants whose arrays are magic squares. Pp. 216-221. J. E. TREVOR. *The Cochlioid.* Pp. 222-227. R. WOODS. *The Correlations between two Variates one of which is normally distributed.* Pp. 227-231. P. R. RIDER. *College Geometry.* Pp. 232-235. N. ALTSHILLER-COURT. *The Arithmetic of Hsia-Hou Yang.* Pp. 235-237. L. VANHÉE. *A Note on Knots.* Pp. 237-239. F. V. MORLEY. *The Definition of "Variable."* Pp. 239-242. A. A. BENNETT.

June, 1924.

A Second Budget of Exercises on Determinants. Pp. 264-274. Sir T. MUIR. *The Trigonometry of Correlation* Pp. 275-280. D. JACKSON. *A Theorem in Thermodynamics.* Pp. 280-283. J. E. TREVOR. *On the Solution of Algebraic Equations with Rational Coefficients.* Pp. 283-287. G. JAMES. *A New Method for the Determination of the Group of Isomorphisms of the Symmetric Group of Degree n.* Pp. 287-289 H. A. BENDER.

Annals of Mathematics. (Princeton, N.J.)

Sept. 1923.

The History of Notations of the Calculus. Pp. 1-46. H. CAJORI. *New Applications of a Fundamental Theorem of Substitution Groups.* Pp. 47-52. G. A. MILLER. *Geodesic Representations between Riemann Spaces.* Pp. 53-56. H. LEVY and A. BRAMLEY. *On the Residues of Figurate Numbers.* Pp. 57-70. O. E. GLENN. *On Symmetric Forms in N Variables.* II. Pp. 71-84. A. DRESDEN. *On an Infinite System of Non-Abelian Groups of Order nm^{n-1}.* Pp. 85-90. W. E. EDINGTON.

Dec. 1923.

A Contribution to the Theory of Closed Chains. Pp. 91-117. A. ARWIN. *On the Characterization of the Set of Points of λ-Continuity.* Pp. 118-122. H. BLUMBERG. *On the Ampère-Cauchy Derived Functions.* Pp. 123-136. H. L. SMITH. *Note on Certain Seminvariants of n-Lines.* Pp. 137-141. L. P. COPELAND. *The Fredholm Theory of Stieltjes Integral Equations.* Pp. 142-155. C. A. FISCHER. *Rectilinear Congruences referred to Special Surfaces.* Pp. 159-180. M. C. FOSTER. *Parallel Propagation of a Vector around an infinitesimal Circuit in an Affine-connected Manifold.* Pp. 181-184. J. L. SYNGE.

Bolletino della Unione Matematica Italiana. (Zanichelli, Bologna.)

April, 1924.

Contributi alla geometria proiettivo-differenziale di una superficie. Pp. 49-56. E. BOMPIANI. *Intorno alla distribuzione della forza elastica nell' imbocco dentato.* Pp. 56-62. O. POMINI. *Per una Revisione delle Leggi della Dinamica.* Pp. 62-65. C. DEL LUNGO. *Sulle derivate di ordine superiore delle funzioni composte.* Pp. 65→. F. SBRANA.

June, 1924.

Contributi alla geometria proiettivo-differenziale di una superficie. Pp. 97-100. E. BOMPIANI. *Su una proposizione fondamentale dell' analisi.* Pp. 103-104. L. TONELLI. *Un teorema sulle funzioni sommabili.* Pp. 104-106. M. PICONE. *Notizie storiche sopra una proprietà dei numeri complessi.* Pp. 107-108. A. AGOSTINI.

Bulletin of the American Mathematical Society. (Lancaster, Pa.)

March-April, 1924.

Problems in Involutorial Transformations in Space. Pp. 101-124. V. SNYDER. *A Generalisation of the Syllogism.* Pp. 125-127. B. A. BERNSTEIN. *On Certain Quinary Quadratic Forms.* Pp. 127-130. E. T. BELL. *The Invariants of Forms under the Binary Linear Homogeneous Group G_6 Modulo 2.* Pp. 131-140. O. E. GLENN. *Ideals and Diophantine Analysis.* Pp. 140—>. G. E. WAHLIN.

May-June, 1924.

A Characterization of Surfaces of Translation. Pp. 231-232. E. P. LANE. *Concerning a Suggested and Discarded Generalization of the Weierstrass Factorization Theorem.* Pp. 233-236. L. L. DINES. *The Class Number Relations implicit in the Disquisitiones Arithmeticae.* Pp. 236-238. E. T. BELL. *Number of Cycles of the same Order in any given Substitution Group.* Pp. 239-246. G. A. MILLER. *Algebras and their Arithmetics.* Pp. 247-257. L. E. DICKSON.

July, 1924.

The Equation of the Eighth Degree. Pp. 301-313. A. B. COBLE. *Group of a Set of Simultaneous Algebraic Equations.* Pp. 314-316. L. WEISNER. *On a Type of Plane Unicursal Curve.* Pp. 317-322. H. HILTON. *Surfaces with Orthogonal Loci of the Centres of Geodesic Curvature of an Orthogonal System.* Pp. 322-327. M. FOSTER. *Quadratic Fields in which Factorisation is always Unique.* Pp. 328-334. L. E. DICKSON. *The Jacobian of a Contact Transformation.* Pp. 335-338. E. F. ALLEN. *Integro-Differential Invariants of One-Parameter Groups of Fredholm Transformations.* Pp. 338-344. A. D. MICHAL. *Reductions of Enumerations in Homogeneous Forms.* Pp. 345-351. E. T. BELL.

Bulletin of the Calcutta Mathematical Society.

Dec. 1923.

Algebra of Polynominals. C. II. Pp. 141-150. N. GHOSE. *On the motion of a viscous liquid between two non-concentric circular cylinders.* Pp. 151-160. S. MITRA. *Transverse vibrations of a thin rotating rod and of a rotating circular ring.* Pp. 161-172. J. GHOSH. *Geometrical representation of analytical equations of conics for complex variables.* Pp. 173-194. M. GHATAK. *The investigations of the forced oscillations set up in an aeroplane by periodic gusts of wind, with special reference to the case of synchrony with the free oscillations.* Pp. 195-218. N. BASU.

March, 1924.

On a Factorable Continuant. Pp. 219-238. S. CHAKRABARTI. *On an application of Bessel Functions to Probability.* Pp. 239-245. A. DATTA. *On Vertex Rings of finite circular section in incompressible fluids.* Pp. 247-254. N. SEN. Note on the *Convergence of Fourier's Series.* Pp. 255-257.

L'Enseignement Mathématique. (Gauthier-Villars.)

April, 1924.

Les Mathématiques en Portugal. Pp. 137-142. F. G. TEXEIRA. *Qu'est-ce que la géométrie analytique?* Pp. 142-148. GINO LORIA. *Fonctions elliptiques et quartiques binodales.* Pp. 148-163. M. WINANTS. *Sur l'intégration de quelques équations différentielles du second ordre.* Pp. 163-174. ABBÉ LAINE. *Sur les invariants d'une équation différentielle.* Pp. 174-185. T. PETROVITCH. *Problèmes du scrutin.* Pp. 187-189. J. AEBLY and D. MIRIMANOFF. *Triangle et Cercle circonscrit.* Pp. 189-193. A. AMIEL. *Tables des fonctions de Bessel.* Pp. 193-203. E. BRASEY.

July, 1924.

Caractères généraux de la pensée scientifique dans la Grèce ancienne. Pp. 257-268. A. REYMOND. *La Pédagogie des Théories d'Einstein.* Pp. 268-286. A. BUHL. *Propriétés involutives d'un système de sections coniques.* Pp. 287-291. P. MERCIER. *Sur les triangles homologiques* Pp. 292-297. H. LEBESGUE. *Déduction géométrique des dérivées supérieures des fonctions circulaires sin x et cos x.* Pp. 297-304. J. ARNOULJEVIC and B. PETRONIEVICS. *Sur l'inversion des produits arithmétiques.* Pp. 305-308. E. T. BELL. *Sur la théorie des équations et des fonctions algébriques d'après l'école géométrique italienne.* Pp. 309-322. F. ENRIQUES.

Gazeta Matematica. (Chibrituri, Bucuresti.)

April, 1924.

Teorema asupra triunghiului. Pp. 281-285. G. C. MOISIL. *Asupra dreptelor lui Simson angente la cercul celor nouă puncte.* Pp. 286-288. A. I. GUŢU.

May, 1924.

Asupra spiralei logaritmice. Pp. 321-324. N. ABRAMESCU. *Asupra Determinantului Simetric Stäng.* Pp. 324-328. TH. ANGHELUŢA.

July, 1924.

Ecuaţiile traectoriei unui proectil in aer. Pp. 406-411. I. LINTES.

August, 1924.

A supra unei teoreme a lui Hermite. Pp. 443-445. N. ABRAMESCU. *O proprietate a indicatorilor de ordin superior.* Pp. 445-448. I. LINTES.

L'Intermédiaire des Mathématiciens. (Gauthier-Villars.)

1922.

Table des Matières et noms d'auteurs.

March-April, 1924.

May-June, 1924.

Jahresbericht der Deutschen Mathematiker-Vereinigung. (Teubner, Leipzig.)

32 Band. 9-12 Heft.

Karl Rohn. Pp. 201-211. F. SCHUR. *M. Noether.* Pp. 211-233. A. BRILL. *Eine Abbildung der binären Trilinearform.* Pp. 234-239. A. DUSCHEK. *Wurzeln der Sätze von der Winkelsumme, den Winkeln am Kreise und dem Kreisviereck in der nichteuclidischen Geometrie.* Pp. 239-251. M. BRUES. *Analysen des drei- und des Vierdimensionalen Vektorraumes.* Pp. 252-271. E. WAELSCH. *Neue Paare von Raumkurven.* Pp. 271-285. BR. SPIEWECK. *Über die Darstellung analytischer Funktionen durch Potenzreihen.* Pp. 286-295. A. OSTROWSKI. *Die vier Umkreise des ebenen Dreiecks.* Pp. 296-299. A. SCHÜLKE. *Analoga zu den Sätzen von Rolle und Bolzano für Komplexe Polynome und Potenzreihen mit Lücken.* Pp. 299-309. M. FEKETE. *Bemerkung zu G. Scheffers Aufsatz: "Eine die konformen Abbildungen kennzeichnende Eigenschaft."* P. 310. E. NEUBAUER.

The Journal of the Indian Mathematical Society. (Varadachari, Madras.)

May, 1924.

Collinearity of Three Points on a Non-singular Cubic (continued). Pp. 165-171. B. S. MADHAVARAO and M. LAKSHMANAMURTHI. *Definitions of Number.* Pp. 172-175. S. V. RAMAMURTY. *Some New Formulae connected with the Working of a Superannuation Fund.* Pp. 176-182. M. VAIDYANATHAN. *Variations of Wind with Height. Coefficient of Turbulence.* Pp. 183-188. R. N. APTE. *Asymptotic Expansions of Certain Series.* Pp. 121-125. S. L. MALURKAR. *Cross-Ratio Invariants.* Pp. 125-126. R. GOPALASWAMI.

Journal of the Mathematical Association of Japan for Secondary Education. (I. Mori, Tokyo Higher Normal School for Girls.)

Dec. 1923. Feb. 1924. June, 1924.

Mathematical Notes. (Lindsay & Co., Edinburgh.)

May, 1924.

Two Extensions of Ceva's Theorem. Pp. 1-2. A. C. AITKEN. *An Infinite Product of Euler* $\left[\frac{3.5.7.11\ldots}{4.4.8.12\ldots}\right]$. P. 3. A. C. AITKEN. *An Elementary Proof of Girard's Theorem* [Every prime of form $4m+1$ is the sum of two squares in one way only, and no prime of form $4m-1$ is a factor of the sum of two squares that are not both multiples of that prime. (1625)]. P. 3. A. C. AITKEN. *Theorem regarding a regular polygon and a circle cutting its sides, with applications.* Pp. 7-10. A. C. RUSSELL. *Direct Proofs of Theorems in Elementary Geometry.* Pp. 10-11. R. F. BLADES. *An Elementary Proof of Feuerbach's Theorem.* Pp. 11-12. K. J. SANJANA.

Nieuw Archief voor Wiskunde. (Noordhoff, Groningen.) Tweede Reeks Deel xiv. Derde Stuk.

Over het aandeel van Isaac Beeckman in de Ontwikkeling der Valwetten. Pp. 186-208. E. J. DIJKSTERHUIS. *Uitbreiding der Formule van Lagrange voor Rekenkundige Reeksen van Hoogere Orde.* Pp. 209-213. F. SCHUH. *Over de Oskulatiesirkel.* Pp. 214-222. H. B. A. BOCKWINKEL. *Over de Korreksieterm in het Teorema van Taylor en de Differentiequotienten (Dividierte Differenzen) van Thiele.* Pp. 223-233. H. B. A. BOCKWINKEL. *Über eine Erweiterung des Cauchy-Goursatschen Integralsatzes.* Pp. 234-239. H. LOOMAN. *Over de Cylinders, behoorende tot een lineair complex van Kwadratische Oppervlakken.* Pp. 240-243. H. J. VAN VEEN. *Over het Differentieren van een Reeks.* Pp. 244-247. H. B. A. BOCKWINKEL.

Periodico di Matematiche. (Zanichelli, Bologna.)

March, 1924.

La regola del poligono delle forze dedotta dalla leva. Pp. 89-91. A. OCCHIALINI. *Sulla trisezione dell' angolo mediante coniche.* Pp. 91-103. E. ZONDADARI. *Come si può rimuovere il paradosso di D'Alembert.* Pp. 103-110. B. CALDONAZZO. *Un semplice dispositivo risolutero per l'equazione trinomia.* Pp. 111-117. O. CHISINI. *Quando, come e da chi ci vennero ricuperati i sette libri delle "Coniche di Apollonio."* Pp. 118-130. E. BORTOLOTTI. *Il "De Viribus Quantitatis" di Luca Pacioli.* Pp. 165-192. A. AGOSTINI.

May, 1924.

Il secondo principio della Termodinamica. Pp. 193-221. E. DANIELE. *Sulle macchine analitiche.* Pp. 222-236. G. TACCHELLA. *Sui libri di testo per l'Aritmetica nelle Scuole elementari.* Pp. 237-242. G. PEANO. *Operazioni aritmetiche e regole algebriche.* Pp. 243-247. G. BURNENGO. *Sul principio d'identità di due polinomi.* Pp. 247-249. E. MACCAFERRI.

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto."—Bacon.

The Mathematical Association, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, Sydney (New South Wales), and Queensland (Brisbane). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. Bell & Sons, Ltd.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) Articles, mainly on subjects within the scope of elementary mathematics;

(2) Notes, generally with reference to shorter and more elegant methods than those in current text-books;

(3) Reviews, written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) Short Notices of books not specially dealt with in the Reviews;

(5) Queries and Answers, on mathematical topics of a general character.

ive
o be